Translog Factor Demand Systems:
The KLEM Model
by

A. Ronald Gallant
Department of Economics
University of North Carolina
Chapel Hill NC 27599-3305 USA

References

Berndt, Ernst R. (1991), *The Practice of Econometrics: Classic and Contemporary*, Addison-Wesley, Reading MA, Chapter 9.

Berndt, Ernst R., and Mohammed S. Khaled (1979), "Parametric Productivity Measurement and Choice Among Flexible Functional Forms," *Journal of Political Economy* 87, 1220-1245.

Berndt, Ernst R., and David O. Wood (1975), "Technology, Prices, and the Derived Demand for Energy," *Review of Economics and Statistics* 57, 259-268.

Berndt, Ernst R., and David O. Wood (1979), "Engineering and Econometric Interpretation of Energy-Capital Complementarity," *American Economic Review* 69, 342-354.

Gallant, A. Ronald (1987), *Nonlinear Statistical Models,* Wiley, New York.

Rossi, Peter E. (1983), *Specification and Analysis of of Econometric Production Models*, Ph.D. dissertation, University of Chicago.

The KLEM Model

- Translog Factor Demand System
  - U.S. Manufacturing, 1957–1971
  - Price and quantity data on capital (K), labor (L), energy (E), materials (M), and ten instrumental variables.
  - Files klem.doc and klem.dat in pub/arg/data at ftp.econ.duke.edu.
- Features
  - A linear multivariate regression if prices exogenous and errors additive.
  - A nonlinear multivariate regression if prices exogenous and errors logistic normal.
  - A linear simultaneous equations system if prices endogenous and errors additive.
  - A nonlinear simultaneous equations system if prices endogenous and errors logistic normal.

Factor Demand Theory in Levels

The producer's cost function $c(p, u)$ gives the minimum cost of producing output u during a given period using inputs $q = (q_1, \ldots, q_N)'$ at prices $p = (p_1, \ldots, p_N)'$.

Linear homogeneity: $\lambda c(p, u) = c(\lambda p, u)$

Constant returns: $c(p, u) = uc(p)$

Shepard's lemma: $q = \frac{\partial}{\partial p} c(p, u)$

Elasticities of substitution: $\sigma_{ij} = \dfrac{c(p,u)\frac{\partial^2}{\partial p_i \partial p_j} c(p,u)}{\frac{\partial}{\partial p_i} c(p,u) \frac{\partial}{\partial p_j} c(p,u)}$

Price elasticities: $\eta_{ij} = \dfrac{\partial \log q_i}{\partial \log p_i} = p_j \sigma_{ij} \dfrac{\frac{\partial}{\partial p_j} c(p,u)}{c(p,u)}$

Factor Demand Theory in Logarithms

The producer's log cost function $g(l, v)$ gives the minimum cost of producing log output $v = \log u$ during a given period using factor cost shares $s = (p_1q_1, \ldots, p_Nq_N)'/(\sum_{i=1}^N p_iq_i)$ at log prices $\ell = (\log p_1, \ldots, \log p_N)'$.

Linear homogeneity: $g(\ell + \tau 1, v) = \tau + g(\ell, v)$
where $1 = (1, \ldots, 1)'$

Constant returns: $g(\ell, v) = v + g(\ell)$

Shepard's lemma: $s = \frac{\partial}{\partial \ell} g(\ell, v)$

Elasticities of subst.: $\Sigma = G^{-1}\Big[\frac{\partial^2 g}{\partial \ell \partial \ell'} + \frac{\partial g}{\partial \ell}\frac{\partial g}{\partial \ell'} - G\Big]G^{-1}$
where $G = \text{diag}(\frac{\partial g}{\partial \ell})$

Price elasticities: $\eta = \Sigma G$

Constant Returns KLEM Translog Model (1)

Factor cost shares: $s = \begin{pmatrix} M_K \\ M_L \\ M_E \\ M_M \end{pmatrix}$

Log factor prices: $\ell = \begin{pmatrix} \log P_K \\ \log P_L \\ \log P_E \\ \log P_M \end{pmatrix}$

Translog log cost function:

$$g(\ell, v) = \alpha \ell + \frac{1}{2}\ell' B \ell + v$$

where

$$\alpha = \begin{pmatrix} \alpha_K \\ \alpha_L \\ \alpha_E \\ \alpha_M \end{pmatrix} \qquad B = \begin{pmatrix} \gamma_{KK} & \gamma_{KL} & \gamma_{KE} & \gamma_{KM} \\ \gamma_{LK} & \gamma_{LL} & \gamma_{LE} & \gamma_{LM} \\ \gamma_{EK} & \gamma_{EL} & \gamma_{EE} & \gamma_{EM} \\ \gamma_{MK} & \gamma_{ML} & \gamma_{ME} & \gamma_{MM} \end{pmatrix}$$

Constant Returns KLEM Translog Model (1)

Because

$$\frac{\partial}{\partial \ell}\ell' B \ell = (B + B')\ell$$

$$\frac{\partial^2}{\partial \ell \partial \ell'}\ell' B \ell = (B + B')$$

and $(B + B')$ is symmetric, we might just as well assume that B is symmetric to start with, which we will.

The Translog factor demand system is

$$s = \alpha + B\ell$$

whose coefficients are subject to the following restrictions:

Linear homogeneity : $\alpha' 1 = 1$, $B1 = 0$
Symmetry : $B = B'$
Adding up: $\alpha' 1 = 1$, $1' B = 0$

some of which are redundant.

KLEM Translog Model, Additive Errors (1)

All one can get out of a deterministic model such as above a specification for some measure of central tendency of some distribution. The simplest assumption is additive measurement error:

$$s_t = \alpha + B\ell_t + e_t$$

where

$$\mathcal{E} e_t = 0$$

$$\mathcal{E} e_s e_t' = \begin{cases} \Omega & s = t \\ 0 & s \neq t \end{cases}$$

$$t = 1, \ldots, n$$

KLEM Translog Model, Additive Errors (2)

This specification should remain invariant to linear transformation. That is, we could just as well choose to analyze the model

$$Ps_t = P\alpha + PB\ell_t + Pe_t$$

for some matrix P; indeed, this is how one usually does generalized least squares in practice. Consider

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We get

$$\begin{aligned} s_{Kt} &= \alpha_K + \gamma_{KK}\ell_{Kt} + \gamma_{KL}\ell_{Lt} + \gamma_{KE}\ell_{Et} + \gamma_{KM}\ell_{Mt} + e_{Kt} \\ s_{Lt} &= \alpha_L + \gamma_{LK}\ell_{Kt} + \gamma_{LL}\ell_{Lt} + \gamma_{LE}\ell_{Et} + \gamma_{LM}\ell_{Mt} + e_{Lt} \\ s_{Et} &= \alpha_E + \gamma_{EK}\ell_{Kt} + \gamma_{EL}\ell_{Lt} + \gamma_{EE}\ell_{Et} + \gamma_{EM}\ell_{Mt} + e_{Et} \end{aligned}$$

$$1 = \sum_{i=K}^{M} s_i = \sum_{i=K}^{M} \alpha_i + \sum_{j=K}^{M} \left(\sum_{i=K}^{M} \gamma_{ij} \right) \ell_{jt} + \sum_{i=K}^{M} e_{it}$$

KLEM Translog Model, Additive Errors (3)

Consider the implications of

$$1 = \sum_{i=K}^{M} \alpha_i + \sum_{j=K}^{M} \left(\sum_{i=K}^{M} \gamma_{ij} \right) \ell_{jt} + \sum_{i=K}^{M} e_{it}$$

Because the LHS is non-random, we have

$$\text{Var}\left(\sum_{i=K}^{M} e_{it} \right) = 0$$

The mean and variance of $\sum_{i=K}^{M} e_{it}$ are therefore both zero, which implies $\sum_{i=K}^{M} e_{it}$ is identically zero for all t. We have now that

$$1 = \sum_{i=K}^{M} \alpha_i + \sum_{j=K}^{M} \left(\sum_{i=K}^{M} \gamma_{ij} \right) \ell_{jt}$$

Because this must hold for all t, we conclude that

$$\sum_{i=K}^{M} \alpha_i = 1 \text{ and } \sum_{i=K}^{M} \gamma_{ij} = 0 \text{ for } j = K, L, E, M,$$

which is the adding up restriction.

KLEM Translog Model, Additive Errors (4)

The KLEM translog factor demand model with additive errors is therefore a three equation linear system

$$\begin{aligned} s_{Kt} &= \alpha_K + \gamma_{KK}\ell_{Kt} + \gamma_{KL}\ell_{Lt} + \gamma_{KE}\ell_{Et} + \gamma_{KM}\ell_{Mt} + e_{Kt} \\ s_{Lt} &= \alpha_L + \gamma_{LK}\ell_{Kt} + \gamma_{LL}\ell_{Lt} + \gamma_{LE}\ell_{Et} + \gamma_{LM}\ell_{Mt} + e_{Lt} \\ s_{Et} &= \alpha_E + \gamma_{EK}\ell_{Kt} + \gamma_{EL}\ell_{Lt} + \gamma_{EE}\ell_{Et} + \gamma_{EM}\ell_{Mt} + e_{Et} \end{aligned}$$

subject to the parametric constraint of adding up

$$\sum_{i=K}^{M} \alpha_i = 1 \text{ and } \sum_{i=K}^{M} \gamma_{ij} = 0 \text{ for } j = K, L, E, M,$$

The parametric constraint is not binding on the parameters of the three equations, which means that one gets $\hat{\alpha}_M$ from estimates of the parameters of the system using the constraint

$$\hat{\alpha}_M = 1 - \hat{\alpha}_K - \hat{\alpha}_L - \hat{\alpha}_E.$$

Similarly

$$\hat{\gamma}_{Mj} = -(\hat{\gamma}_{Kj} + \hat{\gamma}_{Lj} + \hat{\gamma}_{Ej}$$

for $j = K, L, E, M$.

The errors $u_t = (e_{Kt}, e_{Lt}, e_{Et})'$ satisfy

$$\mathcal{E}u_t = 0$$

$$\mathcal{E}u_s u_t' = \begin{cases} \Sigma & s = t \\ 0 & s \neq t \end{cases}$$

$$t = 1, \ldots, n$$

KLEM Model, Logistic Normal Errors (1)

The assumption of additive errors is implausible for a variety of reasons. One of them is that the errors must be heterogeneous due to the restriction that $0 < \mathcal{E}s_{it} + e_{it} < 1$ which implies that

$$-\mathcal{E}s_{it} < e_{it} < 1 - \mathcal{E}s_{it}$$

Thus, the support of the density must depend on ℓ_t which makes an assumption that second moments do not implausible. While one could use the methods of Chapter 2 to correct for heteroskedasticity, starting with plausible assumptions seems preferable.

Rossi(1983) in an extensive empirical investigation determined that the logistic normal assumption is more plausible. That assumption implies

$$\begin{pmatrix} \log(s_K/s_M) - \log[(\mathcal{E}s_K)/(\mathcal{E}s_M)] \\ \log(s_L/s_M) - \log[(\mathcal{E}s_L)/(\mathcal{E}s_M)] \\ \log(s_E/s_M) - \log[(\mathcal{E}s_E)/(\mathcal{E}s_M)] \end{pmatrix} \sim N_3(0, \Sigma)$$

KLEM Model, Logistic Normal Errors (2)

If we put

$$y_t = \begin{pmatrix} \log(s_{Kt}/s_{Mt}) \\ \log(s_{Lt}/s_{Mt}) \\ \log(s_{Et}/s_{Mt}) \end{pmatrix}$$

the model becomes

$$y_t = f[\ell_t, (\alpha, B)] + e_t$$

where

$$f[\ell,(\alpha,B)] = \begin{pmatrix} \log \frac{\alpha_K + \gamma_{KK}\ell_K + \gamma_{KL}\ell_L + \gamma_{KM}\ell_M + \gamma_{KE}\ell_E}{\alpha_M + \gamma_{MK}\ell_K + \gamma_{ML}\ell_L + \gamma_{MM}\ell_M + \gamma_{ME}\ell_E} \\ \log \frac{\alpha_L + \gamma_{LK}\ell_K + \gamma_{LL}\ell_L + \gamma_{LM}\ell_M + \gamma_{LE}\ell_E}{\alpha_M + \gamma_{MK}\ell_K + \gamma_{ML}\ell_L + \gamma_{MM}\ell_M + \gamma_{ME}\ell_E} \\ \log \frac{\alpha_E + \gamma_{EK}\ell_K + \gamma_{EL}\ell_L + \gamma_{EM}\ell_M + \gamma_{EE}\ell_E}{\alpha_M + \gamma_{MK}\ell_K + \gamma_{ML}\ell_L + \gamma_{MM}\ell_M + \gamma_{ME}\ell_E} \end{pmatrix}$$

subject to $1'\alpha = 1$ and $1'B = 0$. This is a three equation nonlinear system.

The nonhomogeneous restriction $1'\alpha = 1$ must be imposed to get identification. Assuming that $\alpha_M \neq 0$, one can impose the normalization rule $\alpha_M = 1$ instead.

KLEM Model, Logistic Normal Errors (3)

One can impose the hypothesis of symmetry and the normalization rule as follows:

$$\begin{array}{lcl@{\qquad}lcl} \alpha_K &=& \theta_1 & \alpha_E &=& \theta_{10} \\ \gamma_{KK} &=& \theta_2 & \gamma_{EK} &=& \theta_4 \\ \gamma_{KL} &=& \theta_3 & \gamma_{KL} &=& \theta_3 \\ \gamma_{KE} &=& \theta_4 & \gamma_{KE} &=& \theta_4 \\ \gamma_{KM} &=& \theta_5 & \gamma_{KM} &=& \theta_5 \\ \alpha_L &=& \theta_6 & \alpha_M &=& 1 \\ \gamma_{LK} &=& \theta_3 & \gamma_{MK} &=& \theta_5 \\ \gamma_{LL} &=& \theta_7 & \gamma_{ML} &=& \theta_9 \\ \gamma_{LE} &=& \theta_8 & \gamma_{ME} &=& \theta_{12} \\ \gamma_{LM} &=& \theta_9 & \gamma_{MM} &=& \theta_{13} \end{array}$$

the response function becomes

$$f(x,\theta) = \begin{pmatrix} \log \frac{\theta_1 + \theta_2 x_1 + \theta_3 x_2 + \theta_4 x_3 + \theta_5 x_4}{1 + \theta_5 x_1 + \theta_9 x_2 + \theta_{12} x_3 + \theta_{13} x_4} \\ \log \frac{\theta_6 + \theta_3 x_1 + \theta_7 x_2 + \theta_8 x_3 + \theta_9 x_4}{1 + \theta_5 x_1 + \theta_9 x_2 + \theta_{12} x_3 + \theta_{13} x_4} \\ \log \frac{\theta_{10} + \theta_4 x_1 + \theta_8 x_2 + \theta_{11} x_3 + \theta_{12} x_4}{1 + \theta_5 x_1 + \theta_9 x_2 + \theta_{12} x_3 + \theta_{13} x_4} \end{pmatrix}$$

where $x = \ell$.

The KLEM Model

- Translog Factor Demand System
  - U.S. Manufacturing, 1957–1971
  - Price and quantity data on capital (K), labor (L), energy (E), materials (M), and ten instrumental variables.
  - Files klem.doc and klem.dat in pub/arg/data at ftp.econ.duke.edu.
- Features
  - A linear multivariate regression if prices exogenous and errors additive.
  - A nonlinear multivariate regression if prices exogenous and errors logistic normal.
  - A linear simultaneous equations system if prices endogenous and errors additive.
  - A nonlinear simultaneous equations system if prices endogenous and errors logistic normal.