THE PENNSYLVANIA STATE UNIVERSITY
Department of Economics

Economics 501 Gallant
Homework 5 Fall 2014
Due Sept. 30

1. At first glance, it is not clear that taking the limit of the expectation of the approximating discrete random variables shown in Figure 2.4 is the same as taking the supremum as required by the definition of expectation. The verification that this is so proceeds as follows. What is shown in the right half of the figure is a lower approximation of X^+ by a discrete random variable $X_N(\omega) = \sum_{i=1}^N x_i I_{F_i}(\omega)$ where the F_i are nonoverlapping intervals. The definition permits F_i to be any measurable set so it is always possible to find a better lower approximation $X^*(\omega) = \sum_{j=1}^J x_j^* I_{F_j^*}(\omega)$, $J \geq N$, where the F_j^* are not so restricted (Why?). Therefore, $\mathcal{E}X^+ = \sup \mathcal{E}X^* \geq \lim_{N\to\infty} \mathcal{E}X_N$ (Why?). By adjusting the x_i of X_N in Figure 2.4 we can obtain an upper approximation $X_N^+(\omega) = \sum_{i=1}^N x_i^+ I_{F_i}(\omega) \geq X^+(\omega)$, where we leave the F_i unchanged. Moreover, for any N and J we have $X_N^+(\omega) > X^*(\omega)$ (Why?). Therefore, $\lim_{N\to\infty} \mathcal{E}X_N^+ \geq \sup \mathcal{E}X^*$ (Why?). But

$$\lim_{N\to\infty} \mathcal{E}X_N^+ = \int_{\frac{1}{2}}^{1} \log[\omega(1-\omega)]\, d\omega = \lim_{N\to\infty} \mathcal{E}X_N$$

(Why?). Therefore, $\mathcal{E}X^+ = \int_{\frac{1}{2}}^{1} \log[\omega(1-\omega)]\, d\omega$ (Why?).

2. Consider the random variable X with density

$$f(x) \quad = \quad \begin{cases} A(4-x^2) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute A.

   (b) Compute the mean of X.

   (c) Compute $P(0 \leq X \leq 1/2)$.

   (d) Compute the variance of X.

   (e) Find the density of the random variable $Y = X^3$.

3. Consider the jointly distributed random variables X and Y with density

$$f(x,y) \quad = \quad \begin{cases} A(x^2+y) & 0 \leq x \leq 5,\, 0 \leq y \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute A.

(b) Compute the marginal density $f(x)$.

(c) Compute the conditional density $f(y|x)$.

(d) Compute the covariance between X and Y .

(e) Are X and Y independent?

(f) Compute $P(1 \leq X \leq 2,\, 1 \leq Y \leq 2)$.