THE PENNSYLVANIA STATE UNIVERSITY
Department of Economics

Economics 501 | Gallant
Homework 6 | Fall 2014
Due Oct. 7

1. For the joint density
$$f_{X,Y}(i,j) = \frac{\binom{8}{i}\binom{8}{j}\binom{64}{20-i-j}}{\binom{80}{20}}$$
with marginal densities
$$f_X(i) = \frac{\binom{8}{i}\binom{72}{20-i}}{\binom{80}{20}} \qquad f_Y(j) = \frac{\binom{8}{j}\binom{72}{20-j}}{\binom{80}{20}}$$
we derived the conditional expectation
$$\mathcal{E}(Y|X)(i) = 8\,\frac{(20-i)}{72}.$$
Direct computation gives
$$\mathcal{E}Y = \sum_{j=0}^{8} j f_Y(j) = 2.$$
Theorem 2.5 states that
$$\mathcal{E}Y = \mathcal{E}[\mathcal{E}(Y|X)] = \sum_{i=0}^{8} \mathcal{E}(Y|X)(i) f_X(i).$$
Compute this sum and compare.

2. Let the random variable Y be defined by
$$Y = \beta_0 + \beta_1 X + \omega,$$
where X and ω are each gotten by coin tossing and are independent. Compute $\mathcal{E}(Y)$, the unconditional expectation of Y. Compute $\mathcal{E}(Y|X)(x)$, the conditional expectation of Y given X.

3. Consider the jointly distributed random variables X and Y with density
$$f(x,y) = \begin{cases} \frac{6}{5}(x^2+y) & 0 \le x \le 1,\, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the marginal density $f(x)$.

(b) Compute the conditional density $f(y|x)$.

(c) Compute $\mathcal{E}(Y|X)(x)$.

4. Show that conditional expectation is an orthogonal projection in the sense that it satisfies the Pythagorean identity

$$\mathcal{E}\left(Y^2\right) = \mathcal{E}\left\{[\mathcal{E}(Y|\mathcal{F}_0)]^2\right\} + \mathcal{E}\left\{[Y - \mathcal{E}(Y|\mathcal{F}_0)]^2\right\}$$

and the random variables $\mathcal{E}(Y|\mathcal{F}_0)$ and $[Y - \mathcal{E}(Y|\mathcal{F}_0)]$ are orthogonal

$$\mathcal{E}\left\{\mathcal{E}(Y|\mathcal{F}_0)\,[Y - \mathcal{E}(Y|\mathcal{F}_0)]\right\} = 0.$$