THE PENNSYLVANIA STATE UNIVERSITY
Department of Economics

Economics 501 Gallant
Homework 9 Fall 2014
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1. Let $Y = \beta_0 + \beta_1 X + E$ where X and E are independent random variables that are distributed $N(\mu_x, \sigma_x^2)$ and $N(0, \sigma_e^2)$, respectively. Compute $\mathcal{E}(Y)$, $\mathcal{E}(Y|X)$, and find the density of Y.

2. Let $y_i = \beta_0 + \beta_1 x_i + e_i$. The random variables e_i, $i = 1, \ldots, n$, are uncorrelated with first moment $\mathcal{E}(e_i) = 0$ and second moment $\mathcal{E}(e_i^2) = \sigma^2$. The x_i, $i = 1, \ldots, n$, are known numbers; they are not random variables. This setup can be written more compactly as $y = X\beta + e$ where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \qquad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \qquad e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

Recall that if U is a random vector and A is a matrix, then $\mathcal{E}(AU) = A\mathcal{E}U$ and $\mathcal{E}(U'AU) = \text{tr}[\mathcal{E}(U'AU)] = \mathcal{E}\text{tr}(U'AU) = \mathcal{E}\text{tr}(AUU') = \text{tr}[\mathcal{E}(AUU')] = \text{tr}[A\mathcal{E}(UU')]$.

(a) Show that $\mathcal{E}ee' = \sigma^2 I$, where I is the n by n identity matrix.

(b) Define $P_X = X(X'X)^{-1}X'$ and define $P_X^\perp = I - P_X$. Note that P_X and $P_X^\perp$ are n by n matrices. Show that $P_X P_X = P_X$, $P_X^\perp P_X^\perp = P_X^\perp$, and $P_X P_X^\perp = 0$.

(c) The least squares estimator is the random variable $\hat{\beta}$ that minimizes the residual sum of squares

$$s(\beta) = (y - X\beta)'(y - X\beta) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2.$$

Show that the first order conditions for minimizing $s(\beta)$ are $(X'X)\hat{\beta} = X'y$.

(d) Part 2c implies that $\hat{\beta} = (X'X)^{-1}X'y$. Show that $s(\hat{\beta}) = y'P_X^\perp y$.

(e) Show that $\mathcal{E}(y'P_X^\perp y) = (n - 2)\sigma^2$.