THE PENNSYLVANIA STATE UNIVERSITY
Department of Economics

Economics 501 | Gallant
Homework 11 | Fall 2014
Nov. 18

1. Let the sample space Ω be the closed interval [0,1], let $P(\cdot)$ be the uniform distribution on Ω, and define a sequence of random variables X_1, X_2, X_3, ... as follows

$$\begin{array}{lll} X_1(\omega) = I_{[0,1]}(\omega) & X_2(\omega) = I_{[0,\frac{1}{2}]}(\omega) & X_4(\omega) = I_{[0,\frac{1}{3}]}(\omega) \\ & X_3(\omega) = I_{(\frac{1}{2},1]}(\omega) & X_5(\omega) = I_{(\frac{1}{3},\frac{2}{3}]}(\omega) \\ & & X_6(\omega) = I_{(\frac{2}{3},1]}(\omega) \end{array}$$

Show that the sequence of random variables $\{X_i\}_{i=1}^{\infty}$ converges in probability to 0 but does not converge almost surely.

Define a sequence of random variables Y_1, Y_2, Y_3, ... as follows

$$\begin{array}{lll} Y_1(\omega) = I_{[0,1]}(\omega) & Y_2(\omega) = I_{[0,\frac{1}{2}]}(\omega) & Y_4(\omega) = I_{[0,\frac{1}{3}]}(\omega) \\ & Y_3(\omega) = I_{(0,\frac{1}{2}]}(\omega) & Y_5(\omega) = I_{(0,\frac{1}{3}]}(\omega) \\ & & Y_6(\omega) = I_{(0,\frac{1}{3}]}(\omega) \end{array}$$

Show that the sequence of random variables $\{Y_i\}_{i=1}^{\infty}$ converges in probability to 0 and converges almost surely to 0.

Derive the density function $f_{X_i}(x)$ of the random variable X_i. Derive the density function $f_{Y_i}(y)$ of the random variable Y_i. Notice that $f_{X_i}(t) = f_{Y_i}(t)$.

Usually all we know about random variables are their distributions. What sample space they are defined on is irrelevant. The Skorokhod representation theorem states that if a sequence of random variables $\{X_i\}_{i=1}^{\infty}$ converges in distribution to a random variable X then it is always possible to find a (possibly different) probability space and random variables $\{Y_i\}_{i=1}^{\infty}$ and Y defined on it with $F_{Y_i}(t) = F_{X_i}(t)$ and $F_Y(t) = F_X(t)$ such that $\{Y_i\}_{i=1}^{\infty}$ converges almost surely to Y. In view of this, the statements such as "convergence in distribution does not imply convergence in probability" or "convergence in probability does not imply convergence almost surely" lose some of their force. The Skorokhod representation theorem says that you can change the rules of the game so that convergence in any one of the three modes implies convergence with respect to the other two.

2. Use the equation

$$\frac{d}{d\theta}\Big[\frac{d}{d\theta}\log f(x|\theta) f(x|\theta)\Big] = \Big[\frac{d^2}{d\theta^2}\log f(x|\theta)\Big] f(x|\theta) + \Big[\frac{d}{d\theta}\log f(x|\theta)\Big]\Big[\frac{d}{d\theta} f(x|\theta)\Big]$$

and an assumption that $(d/d\theta)$ can be passed through the integral sign to obtain

$$\begin{aligned}&\int \frac{d^2}{d\theta^2}\log f(x|\theta) f(x|\theta)\,dx \\ &= \frac{d}{d\theta}\int \frac{d}{d\theta}\log f(x|\theta) f(x|\theta)\,dx - \int \frac{d}{d\theta}\log f(x|\theta)\frac{d}{d\theta} f(x|\theta)\,dx.\end{aligned}$$

Again assuming that $(d/d\theta)$ can be passed through the integral sign show that

$$\int \Big[\frac{d}{d\theta}\log f(x|\theta)\Big] f(x|\theta)\,dx = 0.$$

Show that

$$\int \left[\frac{d}{d\theta}\log f(x|\theta)\right]\left[\frac{d}{d\theta} f(x|\theta)\right] dx = \int \left[\frac{d}{d\theta}\log f(x|\theta)\right]^2 f(x|\theta)\,dx.$$

3. Let $x_1,\ x_2,\ \ldots$ be independently and identically distributed with common density $f(x|\theta^o)$. Use the uniform strong law of large numbers and $\lim_{n\to\infty}\hat{\theta}_n = \theta^o$ a.s. to conclude that the almost sure limit of

$$\hat{\mathcal{I}}_n = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{d}{d\theta}\log f(x_i|\hat{\theta}_n)\right]^2$$

is

$$\mathcal{I} = \int \left[\frac{d}{d\theta}\log f(x|\theta^o)\right]^2 f(x|\theta^o)\,dx.$$