THE PENNSYLVANIA STATE UNIVERSITY
Department of Economics

Economics 501 Gallant
Homework 13 Fall 2014
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1. Let $f(y)$ be a density function and let g(x) be a density function from which it is easy to generate a random draw X on a computer. Suppose that there is a constant c such that $f(s) \leq cg(s)$ for $-\infty < s < \infty$. Consider the following algorithm:

   (a) Draw X from $g(x)$.

   (b) Draw U from the uniform distribution on [0,1].

   (c) If $U \leq \frac{f(X)}{cg(X)}$, then put $Y = X$ and exit.

   (d) If $U > \frac{f(X)}{cg(X)}$, then return to 1a.

   This algorithm is called a rejection algorithm and will generate a random draw Y from the density $f(y)$. Most fast algorithms for simulating from a density are rejection algorithms. With respect to the rejection algorithm, work the following problems.

   (a) Show that $c \geq 1$.

   (b) Show that the rejection algorithm above will generate a random draw Y from the density $f(y)$ by verifying the following. Make sure that you explain carefully why each equality holds by citing the relevant theorem or performing the required algebra and integration in detail.

      i. The probability that both $X \in [a,\, b]$ at step 1a of the algorithm and that exit occurs at step 1c of the algorithm is equal to

      $$\mathcal{E}\left[I_{[a,\,b]}(X) I_{\left[0,\, \frac{f(X)}{cg(X)}\right]}(U)\right]$$

      ii.

      $$\mathcal{E}\left[I_{[a,\,b]}(X) I_{\left[0,\, \frac{f(X)}{cg(X)}\right]}(U)\right] = \mathcal{E}\left\{I_{[a,\,b]}(X)\, \mathcal{E}\left[I_{\left[0,\, \frac{f(X)}{cg(X)}\right]}(U) \,\Big|\, X\right]\right\} = c^{-1} \int_a^b f(x)\, dx$$

iii. The probability that step 1d of the algorithm occurs is equal to

$$\mathcal{E}\left[I_{\left[\frac{f(X)}{cg(X)},1\right]}(U)\right] = \mathcal{E}\left\{\mathcal{E}\left[I_{\left[\frac{f(X)}{cg(X)},1\right]}(U)\,\middle|\,X\right]\right\} = 1 - c^{-1}$$

iv.

$$P[\,a \le Y \le b\,] = \sum_{i=0}^{\infty}\left(1 - c^{-1}\right)^{i} c^{-1}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$

(c) Show that the probability of n rejections (i.e. n occurrences of step 1d of the algorithm) is $c^{-1}\left(1 - c^{-1}\right)^n$ where $n = 0, 1, \ldots$. Show that the expected number of rejections is $c - 1$. (Hint: This is the same problem as computing the expected number of rolls in craps; see negative binomial in the Appendix of the text.)

An implication of $c - 1$ expected rejections, where $c \ge 1$, is that the speed of the algorithm depends on making c as close to 1 as possible. The only way that $c = 1$ is possible is if $g(x) = f(x)$. Thus, the speed of the algorithm depends on making the shape of g as close to the shape of f as possible so that c is as close to 1 as possible.