UNIVERSITY OF NORTH CAROLINA
Department of Economics

Economics 271 Dr. Gallant
Final Exam Fall 1996

1. (15%) Let the sample space be $\Omega = (0,1) \times (0,1)$ and for each $F \subset \Omega$ let $P(A)$ be the area of A. Define two random variables on Ω by

$$X = \omega_1 + \omega_2$$

$$Y = \omega_1 - \omega_2$$

Find the density $f_{X,Y}(x,y)$ of (X,Y).

2. (10%) Let F_i where $i = 1, 2, \ldots$ be an infinite sequence of events from the sample space Ω. Let F be the set of outcomes ω that are in infinitely many of the F_i. Prove that $F = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} F_i$.

3. (10%) Show that if $F_0, F_1, \ldots F_N$ are mutually exclusive and exhaustive, then the collection of all possible unions of the F_i plus the empty set is a σ-algebra.

4. (10%) Let $\mathcal{F}_0$ consist of the empty set plus all possible unions of the mutually exclusive and exhaustive sequence of sets $F_0, \ldots, F_N$. Show that any random variable Z that is $\mathcal{F}_0$-measurable must be of the form $Z(\omega) = \sum_{i=0}^{N} z_i \, I_{F_i}(\omega)$ where the z_i are not necessarily distinct.

5. (10%) Show that conditional expectation is an orthogonal projection in the sense that it satisfies the Pythagorean identity

$$\mathcal{E}\left(Y^2\right) = \mathcal{E}\left\{[\mathcal{E}(Y|\mathcal{F})]^2\right\} + \mathcal{E}\left\{[Y - \mathcal{E}(Y|\mathcal{F})]^2\right\}$$

and the random variables $\mathcal{E}(Y|\mathcal{F})$ and $[Y - \mathcal{E}(Y|\mathcal{F})]$ are orthogonal

$$\mathcal{E}\left\{\mathcal{E}(Y|\mathcal{F})\,[Y - \mathcal{E}(Y|\mathcal{F})]\right\} = 0.$$

6. (15%) Consider the random variable X with density

$$f(x) = \begin{cases} 2A(1-x^2) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Compute A. (ii) Compute the mean of X. (iii) Compute $P(1/2 \leq X \leq 1)$. (iv) Compute the variance of X. (v) Find the density of $Y = X^3$.

7. (15%) Consider the jointly distributed random variables X and Y with density

$$f(x,y) = \begin{cases} A(x^2+y) & 0 \leq x \leq 1,\, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Compute A. (ii) Compute the marginal density $f(x)$. (iii) Compute the conditional density $f(y|x)$. (iv) Compute the covariance between X and Y. (v) Are X and Y independent?

8. (5%) State the definition of almost sure convergence. State the strong law of large numbers. State the definition of convergence in probability. State the weak law of large numbers.

9. (10%) Let the sample space Ω be the closed interval [0,1], let $P(\cdot)$ be the uniform distribution on Ω, and define a sequence of random variables X_1, X_2, X_3, ... as follows

$$\begin{array}{lll} X_1(\omega) = I_{[0,1]}(\omega) & X_2(\omega) = I_{[0,\frac{1}{2}]}(\omega) & X_4(\omega) = I_{[0,\frac{1}{3}]}(\omega) \\ & X_3(\omega) = I_{(\frac{1}{2},1]}(\omega) & X_5(\omega) = I_{(\frac{1}{3},\frac{2}{3}]}(\omega) \\ & & X_6(\omega) = I_{(\frac{2}{3},1]}(\omega) \end{array}$$

Show that the sequence of random variables $\{X_i\}_{i=1}^{\infty}$ converges in probability to 0 but does not converge almost surely.