UNIVERSITY OF NORTH CAROLINA
Department of Economics

Economics 271 — Dr. Gallant
Final Exam — Fall 1998

1. (10%) Let X be a random variable that is neither discrete nor continuous.

   (a) Describe how $\mathcal{E}X$ is defined.

   (b) Compute $\mathcal{E}X$ for

$$X(w) = \begin{cases} \omega & 0 < \omega \le \frac{1}{3} \\ \omega^2 & \frac{1}{3} < \omega \le \frac{2}{3} \\ \omega^3 & \frac{2}{3} < \omega \le 1 \end{cases}$$

   defined on the coin tossing sample space $(\Omega, \mathcal{F}, P)$, where $\Omega = (0, 1]$.

2. (10%) The conditional expectation of Y given X, where Y maps $(\Omega, \mathcal{F})$ into $(\mathcal{Y}, \mathcal{B})$ and X maps $(\Omega, \mathcal{F})$ into $(\mathcal{X}, \mathcal{A})$, is a function $\mathcal{E}(Y|X)(x)$, which maps $(\mathcal{X}, \mathcal{A})$ into $(\mathcal{Y}, \mathcal{B})$, that satisfies the equation

$$\int_F Y(\omega)\, dP(\omega) = \int_F \mathcal{E}(Y|X)\,[X(\omega)]\; dP(\omega)$$

   for every F of the form $F = X^{-1}(A)$ with $A \in \mathcal{A}$. Give the computational formula for $\mathcal{E}(Y|X)(x)$ in these three cases:

   (a) $X(\omega) = \sum_{i=0}^{N} x_i I_{F_i}(\omega)$, where $P(F_0) = 0$ and $P(F_i) > 0$ for $i = 1, \ldots, N$.

   (b) X and Y are continuous random variables with density $f_{X,Y}(x, y)$.

   (c) X and Y are discrete random variables with density $f_{X,Y}(x_i, y_j)$.

3. (5%) State the definition of almost sure convergence. State the strong law of large numbers. State the definition of convergence in probability. State the weak law of large numbers.

4. (10%) Show that the random variables $\mathcal{E}(Y|\mathcal{F}_0)$ and $[Y - \mathcal{E}(Y|\mathcal{F}_0)]$ are orthogonal in the sense that $\mathcal{E}\left\{\mathcal{E}(Y|\mathcal{F}_0)\,[Y - \mathcal{E}(Y|\mathcal{F}_0)]\right\} = 0$.

5. (15%) Consider the jointly distributed random variables X and Y with density

$$f(x,y) = \begin{cases} \frac{6}{5}(x^2+y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Compute the marginal density $f(x)$.

   (b) Compute the conditional density $f(y|x)$.

   (c) Compute $\mathcal{E}(Y|X)(x)$.

   (d) Compute the covariance between X and Y .

   (e) Are X and Y independent?

   (f) Compute $P(1/2 \le X \le 1,\ 1/2 \le Y \le 1)$.

6. (5%) Let X_i be independently and identically distributed with finite variance. Show that $S_n^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$, where $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$, converges almost surely to $\mathrm{Var}(X)$.

7. (10%) Describe verbally, without use of any symbols, how the probability space is constructed for the random experiment of tossing three coins. Include a discussion of the following in your description: how probability is assigned to cubes, which sides of these cubes include their boundaries and which do not, how probability is assigned to finite unions of disjoint cubes, what the collection of all finite unions of disjoint cubes is called, the properties of this collection, how the probability is extended to a collection of sets that includes all countable unions, what this collection is called and its properties.

8. (5%) A typical element ω of the sample space Ω represents what? What is an event? When does an event occur? A typical element of an algebra is what? A typical element of a σ-algebra is what? Is a σ-algebra an algebra?

9. (15%) Let $y_i = \beta_0 + \beta_1 x_i + e_i$. The random variables e_i, $i = 1, \ldots, n$, are uncorrelated with first moment $\mathcal{E}(e_i) = 0$ and second moment $\mathcal{E}(e_i^2) = \sigma^2$. The x_i, $i = 1, \ldots, n$, are known numbers; they are not random variables. This setup can be written more

compactly as $y = X\beta + e$ where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \qquad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \qquad e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

Recall that if U is a random vector and A is a matrix, then $\mathcal{E}(AU) = A\mathcal{E}U$ and $\mathcal{E}(U'AU) = \mathrm{tr}[\mathcal{E}(U'AU)] = \mathcal{E}\mathrm{tr}(U'AU) = \mathcal{E}\mathrm{tr}(AUU') = \mathrm{tr}[\mathcal{E}(AUU')] = \mathrm{tr}[A\mathcal{E}(UU')]$.

(a) Show that $\mathcal{E}ee' = \sigma^2 I$, where I is the n by n identity matrix.

(b) Define $P_X = X(X'X)^{-1}X'$ and define $P_X^{\perp} = I - P_X$. Note that P_X and $P_X^{\perp}$ are n by n matrices. Show that $P_X P_X = P_X$, $P_X^{\perp} P_X^{\perp} = P_X^{\perp}$, and $P_X P_X^{\perp} = 0$.

(c) The least squares estimator is the random variable $\hat{\beta}$ that minimizes the residual sum of squares

$$s(\beta) = (y - X\beta)'(y - X\beta) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2.$$

Show that the first order conditions for minimizing $s(\beta)$ are $(X'X)\hat{\beta} = X'y$.

(d) Part 9c implies that $\hat{\beta} = (X'X)^{-1}X'y$. Show that $s(\hat{\beta}) = y'P_X^{\perp}y$.

(e) Show that $\mathcal{E}(y'P_X^{\perp}y) = (n-2)\sigma^2$.

10. (15%) Let $X_1, \ldots, X_n$ be a random sample from the normal density

$$n(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}.$$

Derive the Wald, Lagrange, and likelihood ratio tests for the hypothesis

$$H: \mu = \mu^o \quad \text{against} \quad A: \mu \neq \mu^o.$$