Internet Appendix to "Does Smooth Ambiguity Matter for Asset Pricing?"

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1 Numerical methods

We use the collocation projection method with Chebyshev polynomials to solve asset pricing models in the paper. See Judd (1992) for an introduction to projection methods and Pohl et al. (2018) for applications to solving models with long-run risks.

We solve each model in two steps. In the first step, we use the projection method to solve the functional equation for the value function $V_t(C)$ to obtain the wealth-consumption ratio. Suppose that the vector of state variables for a model is denoted by z_t (e.g., $z_t = \{\pi_t\}$ in model AAMS). By homogeneity, we have $V_t(C) = C_t G(z_t)$ where $G(z_t)$ is a function to be determined. As shown by Epstein and Zin (1989), the wealth-consumption ratio W_t/C_t is given by

$$\frac{W_t}{C_t} = \frac{1}{1-\beta}\left(\frac{V_t}{C_t}\right)^{1-\frac{1}{\psi}}.$$

In the second step, we apply the projection method to solve the Euler equation to obtain the price-dividend ratio, given that we can determine the SDF $M_{t,t+1}$ from the solution in the first step. We denote the current state of the economy by z and the next period's state by z'.

1.1 Solving the AAMS Model

This model is developed by Ju and Miao (2012). See "Ambiguity, Learning, and Asset Returns: Technical Appendix" for details about the numerical method.

The functional equation for $G(\pi)$ implied by the generalized recursive smooth ambiguity utility function is given by

$$G(\pi) = \left[(1-\beta) + \beta\left(\mathbb{E}\left[\left(\mathbb{E}\left[G(\pi')^{1-\gamma}\exp\left((1-\gamma)\Delta c(s')\right)\Big| s'\right]\right)^{\frac{1-\eta}{1-\gamma}}\Big|\pi\right]\right)^{\frac{1-1/\psi}{1-\eta}}\right]^{\frac{1}{1-1/\psi}}. \quad (1)$$

The intertemporal marginal rate of substitution (or stochastic discount factor) is given by

$$\begin{aligned} M(\pi', s'|\pi) &= \beta\exp\left(-\frac{1}{\psi}\Delta c(s')\right)\left(\frac{G(\pi')\exp(\Delta c(s'))}{\mathcal{R}(G(\pi')\exp(\Delta c(s'))|\pi)}\right)^{\frac{1}{\psi}-\gamma} \\ &\times\left(\frac{\left(\mathbb{E}\left[G(\pi')^{1-\gamma}\exp\left((1-\gamma)\Delta c(s')\right)\Big| s',\pi\right]\right)^{\frac{1}{1-\gamma}}}{\mathcal{R}(G(\pi')\exp(\Delta c(s'))|\pi)}\right)^{-(\eta-\gamma)}. \end{aligned}$$

The price-dividend ratio $\varphi(\pi)$ satisfies the Euler equation

$$\varphi(\pi) = \mathbb{E}\left[M(\pi', s'|\pi)\left(1+\varphi(\pi')\right)\exp\left(\Delta d(s')\right)\Big|\pi\right]. \quad (2)$$

The laws of motion of consumption and dividend growth are

$$\begin{aligned}\Delta c(s) &= \mu(s) + \sigma_c \epsilon_c, \quad \epsilon_c \sim N(0,1) \\ \Delta d(s) &= \lambda \Delta c(s) + g_d + \tilde{\sigma}_d \epsilon_d, \ \epsilon_d \sim N(0,1)\end{aligned}$$

where the transition probabilities are

$$\Pr\left(s' = l | s = l\right) = p_{ll}, \ \Pr\left(s' = h | s = h\right) = p_{hh}$$

and ϵ_c and ϵ_d are two independent innovation shocks.

The (nonlinear) law of motion of the state variable π is

$$\pi' = \frac{p_{hh} f\left(\Delta c\left(s'\right) | s' = h\right) \pi + \left(1 - p_{ll}\right) f\left(\Delta c\left(s'\right) | s' = l\right)(1 - \pi)}{f\left(\Delta c\left(s'\right) | s' = h\right) \pi + f\left(\Delta c\left(s'\right) | s' = l\right)(1 - \pi)}.$$

We approximate the solution functions $G(\pi)$ and $\varphi(\pi)$ by Chebyshev polynomials, namely,

$$\hat{G}\left(\pi; \boldsymbol{a}^G\right) = \sum_{k=0}^{n_\pi} a_k^G T_k\left(t_\pi\right), \quad \hat{\varphi}\left(\pi; \boldsymbol{a}^\varphi\right) = \sum_{k=0}^{n_\pi} a_k^\varphi T_k\left(t_\pi\right)$$

where $T_k : [-1, 1] \to \mathbb{R}, \ k = 0, 1, ..., n_\pi$ are Chebyshev polynomials and the transformation of the argument for the polynomial is given by

$$t_\pi = 2\left(\frac{\pi - \pi_{\min}}{\pi_{\max} - \pi_{\min}}\right) - 1$$

with $\pi_{\min} = 0$ and $\pi_{\max} = 1$. To implement the collocation method, we solve the two functional equations (1) and (2) on a grid of π obtained by applying the inverse of the transformation to the $n_\pi + 1$ zeros of the Chebyshev polynomial $T_{n_\pi+1}$.

Equations (1) and (2) define two residual functions that are to be minimized sequentially by choosing the coefficients $\boldsymbol{a}^G$ and $\boldsymbol{a}^\varphi$. The collocation projection method leads to two square systems of nonlinear equations, which can be solved with a nonlinear equations solver (e.g., Powell's hybrid algorithm). Because the underlying innovation shocks are Gaussian, we use Gauss-Hermite quadrature to calculate conditional expectations in the residual functions.

1.2 Solving the AAMSTV Model

Compared to AAMS, the AAMSTV model has one additional state variable s_t^σ indicating the volatility state. It follows that $G(\pi, s^\sigma)$ satisfies the equation

$$G\left(\pi, s^\sigma\right) = \left[(1-\beta) + \beta\left(\mathbb{E}\left[\left(\mathbb{E}\left[G\left(\pi', s^{\sigma\prime}\right)^{1-\gamma} \exp\left((1-\gamma)\Delta c\left(s^{\mu\prime}, s^{\sigma\prime}\right)\right) \middle| s^{\mu\prime}, s^\sigma\right]\right)^{\frac{1-\eta}{1-\gamma}} \middle| \pi\right]\right)^{\frac{1-1/\psi}{1-\eta}}\right]^{\frac{1}{1-1/\psi}}.$$

The SDF and Euler equation are given by

$$\begin{aligned} M\left(\pi', s^{\sigma\prime}, s^{\mu\prime}|\pi, s^{\sigma}\right) &= \beta \exp\left(-\frac{1}{\psi}\Delta c\left(s^{\mu\prime}, s^{\sigma\prime}\right)\right)\left(\frac{G\left(\pi', s^{\sigma\prime}\right)\exp\left(\Delta c\left(s^{\mu\prime}, s^{\sigma\prime}\right)\right)}{\mathcal{R}\left(G\left(\pi', s^{\sigma\prime}\right)\exp\left(\Delta c\left(s^{\mu\prime}, s^{\sigma\prime}\right)\right)\middle|\,\pi, s^{\sigma}\right)}\right)^{\frac{1}{\psi}-\gamma} \\ &\times\left(\frac{\left(\mathbb{E}\left[G\left(\pi', s^{\sigma\prime}\right)^{1-\gamma}\exp\left(\left(1-\gamma\right)\Delta c\left(s^{\mu\prime}, s^{\sigma\prime}\right)\right)\middle|\, s^{\mu\prime}, \pi, s^{\sigma}\right]\right)^{\frac{1}{1-\gamma}}}{\mathcal{R}\left(G\left(\pi', s^{\sigma\prime}\right)\exp\left(\Delta c\left(s^{\mu\prime}, s^{\sigma\prime}\right)\right)\middle|\,\pi, s^{\sigma}\right)}\right)^{-(\eta-\gamma)} \end{aligned}$$

and

$$\varphi\left(\pi, s^{\sigma}\right) = \mathbb{E}\left[M\left(\pi', s^{\sigma\prime}, s^{\mu\prime}|\pi, s^{\sigma}\right)\left(1+\varphi\left(\pi', s^{\sigma\prime}\right)\right)\exp\left(\Delta d\left(s^{\mu\prime}, s^{\sigma\prime}\right)\right)\middle|\,\pi, s^{\sigma}\right].$$

The laws of motions for consumption and dividend growth are

$$\begin{aligned} \Delta c\left(s^{\mu}, s^{\sigma}\right) &= \mu\left(s^{\mu}\right) + \sigma\left(s^{\sigma}\right)\epsilon_c, \quad \epsilon_c \sim N\left(0,1\right) \\ \Delta d\left(s^{\mu}, s^{\sigma}\right) &= \lambda\Delta c\left(s^{\mu}, s^{\sigma}\right) + g_d + \tilde{\sigma}_d\epsilon_d,\ \epsilon_d \sim N\left(0,1\right) \end{aligned}$$

where the transition probabilities for the two independent Markov chains of s^{μ} and s^{σ} are given by

$$\begin{aligned} \Pr\left(s^{\sigma\prime} = l|s^{\sigma} = l\right) &= p_{ll}^{\sigma},\ \Pr\left(s^{\sigma\prime} = h|s^{\sigma} = h\right) = p_{hh}^{\sigma} \\ \Pr\left(s^{\mu\prime} = l|s^{\mu} = l\right) &= p_{ll}^{\mu},\ \Pr\left(s^{\mu\prime} = h|s^{\mu} = h\right) = p_{hh}^{\mu} \end{aligned}$$

The law of motion of the state variable π is given by the Bayes' rule

$$\pi' = \frac{p_{hh}^{\mu} f\left(\Delta c\left(s^{\mu\prime}, s^{\sigma\prime}\right)|s^{\mu\prime} = h, s^{\sigma\prime}\right)\pi + \left(1-p_{ll}^{\mu}\right) f\left(\Delta c\left(s^{\mu\prime}, s^{\sigma\prime}\right)|s^{\mu\prime} = l, s^{\sigma\prime}\right)\left(1-\pi\right)}{f\left(\Delta c\left(s^{\mu\prime}, s^{\sigma\prime}\right)|s^{\mu\prime} = h, s^{\sigma\prime}\right)\pi + f\left(\Delta c\left(s^{\mu\prime}, s^{\sigma\prime}\right)|s^{\mu\prime} = l, s^{\sigma\prime}\right)\left(1-\pi\right)}$$

We approximate the solutions to $G\left(\pi, s^{\sigma}\right)$ and $\varphi\left(\pi, s^{\sigma}\right)$ by Chebyshev polynomials as

$$\begin{aligned} \hat{G}\left(\pi, s^{\sigma} = l; \boldsymbol{a}_l^G\right) &= \sum_{k=0}^{n_\pi} a_{k,l}^G T_k\left(t_\pi\right),\ \ \hat{G}\left(\pi, s^{\sigma} = h; \boldsymbol{a}_h^G\right) = \sum_{k=0}^{n_\pi} a_{k,h}^G T_k\left(t_\pi\right) \\ \hat{\varphi}\left(\pi, s^{\sigma} = l; \boldsymbol{a}_l^{\varphi}\right) &= \sum_{k=0}^{n_\pi} a_{k,l}^{\varphi} T_k\left(t_\pi\right),\ \ \hat{\varphi}\left(\pi, s^{\sigma} = h; \boldsymbol{a}_h^{\varphi}\right) = \sum_{k=0}^{n_\pi} a_{k,h}^{\varphi} T_k\left(t_\pi\right) \end{aligned}$$

i.e., we seek four sets of coefficients $\left(\boldsymbol{a}_l^G, \boldsymbol{a}_h^G, \boldsymbol{a}_l^{\varphi}, \boldsymbol{a}_h^{\varphi}\right)$ that minimize the residual functions.

1.3 Solving the AALRRSV Model

We consider the long-run risk model

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_{t+1} + \sigma_t \epsilon_{c,t+1} \\
\Delta d_{t+1} &= \mu_d + \lambda x_{t+1} + \phi_d \sigma_t \epsilon_{d,t+1} \\
x_{t+1} &= \rho_x x_t + \phi_x \sigma_t \epsilon_{x,t+1} \\
\sigma_{t+1}^2 &= \mu_s^2 + \rho_s \left(\sigma_t^2 - \mu_s^2\right) + \sigma_w \epsilon_{w,t+1} \\
\epsilon_{c,t+1}, \epsilon_{d,t+1}, \epsilon_{x,t+1}, \epsilon_{w,t+1} &\sim i.i.d. N(0,1).
\end{aligned}$$

where the long-run risk component x_t is unobservable. We define $\hat{x}_{t+1|t} = E\left[x_{t+1}|\mathcal{I}_t\right]$ and $\nu_{t+1|t} = E\left[\left(x_{t+1} - \hat{x}_{t+1|t}\right)^2 |\mathcal{I}_t\right]$ where $\mathcal{I}_t$ denotes available information at time t. It immediately follows that

$$\hat{x}_{t+1|t} = \rho_x \hat{x}_t, \text{ and } \nu_{t+1|t} = \rho_x^2 \nu_t + \phi_x^2 \sigma_t^2.$$

The Kalman filter implies the following updating equations

$$\begin{aligned}
\hat{x}_{t+1} &= \hat{x}_{t+1|t} + \nu_{t+1|t} \begin{bmatrix} 1 & \lambda \end{bmatrix} F_{t+1|t}^{-1} \begin{bmatrix} v_{t+1|t}^c \\ v_{t+1|t}^d \end{bmatrix} \\
\nu_{t+1} &= \nu_{t+1|t} - \nu_{t+1|t}^2 \begin{bmatrix} 1 & \lambda \end{bmatrix} F_{t+1|t}^{-1} \begin{bmatrix} 1 & \lambda \end{bmatrix}'
\end{aligned}$$

where $F_{t+1|t}$ is given by

$$F_{t+1|t} = \begin{bmatrix} \nu_{t+1|t} + \sigma_t^2 & \lambda \nu_{t+1|t} \\ \lambda \nu_{t+1|t} & \lambda^2 \nu_{t+1|t} + \phi_d^2 \sigma_t^2 \end{bmatrix}$$

and the innovation vector $\begin{bmatrix} v_{t+1|t}^c & v_{t+1|t}^d \end{bmatrix}$ is given by

$$\begin{bmatrix} v_{t+1|t}^c \\ v_{t+1|t}^d \end{bmatrix} = \begin{bmatrix} \Delta c_{t+1} - \mu_c - \rho_x \hat{x}_t \\ \Delta d_{t+1} - \mu_d - \lambda \rho_x \hat{x}_t \end{bmatrix}.$$

Expressed as an intertemporal equation, the solution function $G(\hat{x}, \nu, \sigma)$ satisfies

$$G(\hat{x}, \nu, \sigma) = \left[(1-\beta) + \beta \left(\mathbb{E}\left[\left(\mathbb{E}\left[G\left(\hat{x}', \nu', \sigma'\right)^{1-\gamma} \exp\left((1-\gamma)\Delta c\left(x'\right)\right)\Big| x, \sigma\right]\right)^{\frac{1-\eta}{1-\gamma}} \Big| \hat{x}, \nu\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\eta}}\right]^{\frac{1}{1-\frac{1}{\psi}}}.$$

The SDF and Euler equation are given by

$$\begin{aligned} M\left(x', x, \hat{x}', \nu', \sigma' | \hat{x}, \nu, \sigma\right) &= \beta \exp\left(-\frac{1}{\psi}\Delta c\left(x'\right)\right)\left(\frac{G\left(\hat{x}', \nu', \sigma'\right)\exp\left(\Delta c\left(x'\right)\right)}{\mathcal{R}\left(G\left(\hat{x}', \nu', \sigma'\right)\exp\left(\Delta c\left(x'\right)\right)\middle| \hat{x}, \nu, \sigma\right)}\right)^{\frac{1}{\psi}-\gamma} \\ &\times \left(\frac{\left(\mathbb{E}\left[G\left(\hat{x}', \nu', \sigma'\right)^{1-\gamma}\exp\left(\left(1-\gamma\right)\Delta c\left(x'\right)\right)\middle| x, \hat{x}, \nu, \sigma\right]\right)^{\frac{1}{1-\gamma}}}{\mathcal{R}\left(G\left(\hat{x}', \nu', \sigma'\right)\exp\left(\Delta c\left(x'\right)\right)\middle| \hat{x}, \nu, \sigma\right)}\right)^{-(\eta-\gamma)} \end{aligned}$$

and

$$\varphi\left(\hat{x}, \nu, \sigma\right) = \mathbb{E}\left[M\left(x', x, \hat{x}', \nu', \sigma' | \hat{x}, \nu, \sigma\right)\left(1+\varphi\left(\hat{x}', \nu', \sigma'\right)\right)\exp\left(\Delta d\left(x'\right)\right)\middle| \hat{x}, \nu, \sigma\right].$$

We approximate the solution functions $G\left(\hat{x}, \nu, \sigma\right)$ and $\varphi\left(\hat{x}, \nu, \sigma\right)$ by three-dimensional product Chebyshev polynomials, namely,

$$\begin{aligned} \hat{G}\left(\hat{x}, \nu, \sigma; \boldsymbol{a}^{G}\right) &= \sum_{k_{\hat{x}}=0}^{n_{\hat{x}}}\sum_{k_{\nu}=0}^{n_{\nu}}\sum_{k_{\sigma}=0}^{n_{\sigma}} a^{G}_{k_{\hat{x}}k_{\nu}k_{\sigma}} T_{k_{\hat{x}}}\left(t_{\hat{x}}\right) T_{k_{\nu}}\left(t_{\nu}\right) T_{k_{\sigma}}\left(t_{\sigma}\right) \\ \hat{\varphi}\left(\hat{x}, \nu, \sigma; \boldsymbol{a}^{\varphi}\right) &= \sum_{k_{\hat{x}}=0}^{n_{\hat{x}}}\sum_{k_{\nu}=0}^{n_{\nu}}\sum_{k_{\sigma}=0}^{n_{\sigma}} a^{\varphi}_{k_{\hat{x}}k_{\nu}k_{\sigma}} T_{k_{\hat{x}}}\left(t_{\hat{x}}\right) T_{k_{\nu}}\left(t_{\nu}\right) T_{k_{\sigma}}\left(t_{\sigma}\right). \end{aligned}$$

In constructing Chebyshev polynomials as basis functions, we obtain the lower and upper bounds for each state variable by simulations. Because $x_t \sim N\left(\hat{x}_t, \nu_t\right)$, we use Gauss-Hermite quadrature to compute the conditional expectation involving state x_t. To compute conditional expectations with respect to the underlying shocks $\left(\epsilon_c, \epsilon_d, \epsilon_x, \epsilon_\sigma\right)$, we apply the monomial method with degree 5, see Judd (1999) for details of the monomial method. If the dimension of underlying shocks is d, the monomial method requires $2d^2+1$ points to compute an expectation, whereas Gauss-Hermite quadrature requires N^d nodes with N being the number of nodes in one dimension. When the dimension of underlying shocks is large, the monomial method is much more efficient than quadrature methods. This gain in efficiency is particularly important for our structural estimation. A number of simulations suggest that for our model the monomial method yields accurate results compared with Gauss-Hermite quadrature.

To implement the collocation method, we solve the two square systems of nonlinear equations derived from equilibrium conditions on a grid of dimension $\left(n_{\hat{x}}+1\right)\times\left(n_\nu+1\right)\times\left(n_\sigma+1\right)$ for the state variables. The grid is constructed from zeros of Chebyshev polynomials of all state variables.

An alternative approach is to discretize the AR(1) process of σ_t^2 into a $n-$state Markov chain by the method developed in Tauchen (1986). Caldara et al. (2012) adopt this approach to solve DSGE models with recursive preferences and stochastic volatility. To avoid negative volatility states in the Markov chain, we keep positive values only and normalize transition probabilities accordingly. As such, given each volatility state σ_i, the solution functions $G\left(\hat{x}, \nu, \sigma_i\right)$ and $\varphi\left(\hat{x}, \nu, \sigma_i\right)$ can be approximated by two-dimensional product Chebyshev polynomials in $\hat{x}$ and ν. Through simulations, we find that this approach yields results that are close to the approximation with three-dimensional product Chebyshev polynomials.

1.4 Solving the EZLRRSV Model

The laws of motion of Δc, Δd, x and σ are given by the long-run risk model

$$\begin{aligned}
\Delta c &= \mu_c + x + \sigma\epsilon_c \\
\Delta d &= \mu_d + \lambda x + \phi_d \sigma \epsilon_d + \phi_c \sigma \epsilon_c \\
x' &= \rho_x x + \phi_x \sigma \epsilon_x \\
\sigma^{2\prime} &= \mu_s^2 + \rho_s \left(\sigma^2 - \mu_s^2\right) + \sigma_w \epsilon_w \\
\epsilon_c, \epsilon_d, \epsilon_x, \epsilon_w &\sim i.i.d. N\left(0,1\right).
\end{aligned}$$

The solution function $G\left(x,\sigma^2\right)$ satisfies

$$G\left(x,\sigma^2\right) = \left[\left(1-\beta\right) + \beta\left(\mathbb{E}\left[G\left(x',\sigma^{2\prime}\right)^{1-\gamma}\exp\left(\left(1-\gamma\right)\Delta c\right)\Big|\, x,\sigma^2\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right]^{\frac{1}{1-\frac{1}{\psi}}}$$

The SDF and Euler equation are given by

$$M\left(x',\sigma^{2\prime}|x,\sigma^2\right) = \beta\exp\left(-\frac{1}{\psi}\Delta c\right)\left(\frac{G\left(x',\sigma^{2\prime}\right)\exp\left(\Delta c\right)}{\mathcal{R}\left(G\left(x',\sigma^{2\prime}\right)\exp\left(\Delta c\right)\big|\, x,\sigma^2\right)}\right)^{\frac{1}{\psi}-\gamma}$$

and

$$\varphi\left(x,\sigma^2\right) = \mathbb{E}\left[M\left(x',\sigma^{2\prime}|x,\sigma^2\right)\left(1+\varphi\left(x,\sigma^2\right)\right)\exp\left(\Delta d\right)\big|\, x,\sigma^2\right]$$

We approximate the solution functions $G\left(x,\sigma^2\right)$ and $\varphi\left(x,\sigma^2\right)$ by two-dimensional product Chebyshev polynomials in x and σ^2:

$$\begin{aligned}
\hat{G}\left(x,\sigma^2;\boldsymbol{a}^G\right) &= \sum_{k_x=0}^{n_x}\sum_{k_\sigma=0}^{n_\sigma} a^G_{k_x k_\sigma} T_{k_x}\left(t_x\right) T_{k_\sigma}\left(t_\sigma\right) \\
\hat{\varphi}\left(x,\sigma^2;\boldsymbol{a}^\varphi\right) &= \sum_{k_x=0}^{n_{\hat{x}}}\sum_{k_\sigma=0}^{n_\sigma} a^\varphi_{k_x k_\sigma} T_{k_x}\left(t_x\right) T_{k_\sigma}\left(t_\sigma\right).
\end{aligned}$$

2 Numerical accuracy

We use the method proposed by Judd (1992) to assess numerical accuracy of our numerical solutions. The numerical accuracy check is through computing the Euler equation error. Previous studies such as Guerrieri and Iacoviello (2015) and Collard, Mukerji, Sheppard, and Tallon (2018) rely on this approach to assess the accuracy of their numerical solutions. Note that instead of computing the Euler equation error implied by calibrated parameters as previous studies do, we compute the error based on the MCMC chain of parameter estimates for each asset pricing model. For each model, we compute several metrics of the error on a chain of estimates (12,000 sets of estimates) obtained from the GSM Bayesian estimation.

For the AAMS model, the Euler equation errors defined on the dividend claim and consumption claim are respectively given by

$$
\begin{aligned}
EulerErr_t^D &= \frac{-C_t + \left(\mathbb{E}_t\left[\beta C_{t+1}^{-1/\psi}\left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})}\right)^{1/\psi-\gamma}\left(\frac{\left(\mathbb{E}_{\{s_{t+1},t\}}[V_{t+1}^{1-\gamma}]\right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_t(V_{t+1})}\right)^{-(\eta-\gamma)}\frac{\frac{P_{t+1}}{D_{t+1}}+1}{\frac{P_t}{D_t}}\frac{D_{t+1}}{D_t}\right]\right)^{-\psi}}{C_t} \\
EulerErr_t^C &= \frac{-C_t + \left(\mathbb{E}_t\left[\beta C_{t+1}^{-1/\psi}\left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})}\right)^{1/\psi-\gamma}\left(\frac{\left(\mathbb{E}_{\{s_{t+1},t\}}[V_{t+1}^{1-\gamma}]\right)^{\frac{1}{1-\gamma}}}{\mathcal{R}_t(V_{t+1})}\right)^{-(\eta-\gamma)}\frac{\frac{W_{t+1}}{C_{t+1}}}{\frac{W_t}{C_t}-1}\frac{C_{t+1}}{C_t}\right]\right)^{-\psi}}{C_t}.
\end{aligned}
$$

The errors are defined in a similar way for other models including AAMSTV, AALRRSV, EZLRRSV and EZMS. The differences are only with regard to the SDF and conditioning state variables. This measure is expressed as a fraction of consumption goods, namely the residual of the Euler equation normalized by consumption. $EulerErr_t^D$ ($EulerErr_t^C$) quantifies the error the agent would commit if he use the approximate solution for the price of the dividend (consumption) claim to decide on marginal investment.

Following Judd (1992), we consider several metrics of the error to evaluate numerical accuracy:

$$
\begin{aligned}
\mathcal{E}_1^D &= \log_{10}\left(\mathbb{E}\left[\left|EulerErr_t^D\right|\right]\right),\ \mathcal{E}_2^D = \log_{10}\left(\mathbb{E}\left[\left(EulerErr_t^D\right)^2\right]\right) \\
\mathcal{E}_1^C &= \log_{10}\left(\mathbb{E}\left[\left|EulerErr_t^C\right|\right]\right),\ \mathcal{E}_2^C = \log_{10}\left(\mathbb{E}\left[\left(EulerErr_t^C\right)^2\right]\right).
\end{aligned}
$$

We report the mean, 5 percentile and 95 percentile of each metric evaluated on the MCMC chains of estimates. It is important to note that we compute the Euler equation error outside the grid that we use to implement the collocation projection method. This is done because we want to assess whether our approximate solutions perform well for simulated data under each model, and because in the GSM Bayesian estimation we use the simulated data to find the mapping recovery from structural parameters to the auxiliary model parameters. We report all measures in $\log_{10}$ terms. For example, a value of $\mathcal{E}_1^D$ equal to -3 suggests that if an agent relies on the approximate solution of the price of the dividend claim, he would expect to make a mistake of 1 dollar for each \$1000 risky investment. The economic interpretation is similar for $\mathcal{E}_1^C$. The metric $\mathcal{E}_2^{D(C)}$ measures the quadratic average of the error. Results reported in Table 1 show that our approximate solutions are accurate.

Table 1: **Numerical accuracy: Euler errors**

Model	$\mathcal{E}_1^D$	$\mathcal{E}_2^D$	$\mathcal{E}_1^C$	$\mathcal{E}_2^C$
AAMS				
Mean	-2.654	-5.281	-3.656	-7.282
95%	-2.179	-4.334	-3.230	-6.437
5%	-3.256	-6.480	-4.233	-8.420
AAMSTV				
Mean	-2.594	-4.940	-4.100	-7.985
95%	-1.826	-3.282	-3.165	-6.120
5%	-3.918	-7.679	-5.612	-11.072
AALRRSV				
Mean	-2.207	-4.083	-4.093	-7.550
95%	-1.633	-2.956	-2.983	-5.551
5%	-2.626	-4.951	-4.932	-9.297
EZLRRSV				
Mean	-2.877	-5.387	-2.820	-5.255
95%	-2.724	-5.066	-2.690	-4.966
5%	-3.126	-5.880	-3.044	-5.708
EZMS				
Mean	-3.751	-7.335	-4.572	-8.987
95%	-3.251	-6.026	-3.979	-7.834
5%	-4.621	-9.077	-5.444	-10.737
EZMSTV				
mean	-4.443	-8.710	-5.354	-10.522
95%	-3.304	-6.532	-3.956	-7.824
5%	-5.342	-10.468	-6.849	-13.517

Table 2: **Prior: the AAMS Model**

Parameter	Min	Max	μ	σ
β	0.9	0.995	0.985	0.005
γ	0.1	100	5	1
ψ	0.1	10	1.5	0.2
η	γ	200	8.87	2
p_{ll}	0.2	0.999	0.516	0.13
p_{hh}	0.2	0.999	0.978	0.24
μ_l	-0.08	0.00	-0.0678	0.017
μ_h	0.00	0.08	0.022	0.006
λ	1	6	2.74	0.8
σ_c	0.004	0.06	0.03	0.0075
σ_d	0.03	0.3	0.13	0.03

3 Priors on structural parameters

We report support conditions (Min and Max), prior location and scale parameters for structural parameters in models AAMS, AAMSTV, AALRRSV and EZLRRSV.[1] For each model, the prior is the combination of the product of independent normal density functions and support conditions. The product of independent normal density functions is given by

$$\xi(\theta) = \prod_{i=1}^{\tilde{n}} N\left[\theta_i | \left(\theta_i^*, \sigma_\theta^2\right)\right]$$

where $\tilde{n}$ denotes the number of parameters. Because this prior is intersected with support conditions that are not all of product form, and because a support condition that rejects parameter values in the MCMC chain implies extreme parameter values such that the solution method fails, this is not an independence prior.

[1] The EZMS (EZMSTV) model has the same prior on parameters as the AAMS (AAMSTV) model does except for the absence of the ambiguity aversion parameter η.

Table 3: **Prior: the AAMSTV Model**

Parameter	Min	Max	μ	σ
β	0.9	0.995	0.985	0.005
γ	0.1	100	5	1
ψ	0.1	10	1.5	0.2
η	γ	200	8.87	2
p_{ll}	0.2	0.999	0.516	0.13
p_{hh}	0.2	0.999	0.978	0.24
μ_l	-0.08	0.00	-0.0678	0.017
μ_h	0.00	0.08	0.022	0.006
p_{ll}^{σ}	0.2	0.999	0.85	0.07
p_{hh}^{σ}	0.2	0.999	0.85	0.07
σ_l	0.004	0.06	0.015	0.0038
σ_h	0.004	0.06	0.03	0.0075
λ	1	6	2.74	0.8
σ_d	0.03	0.3	0.13	0.03

Table 4: **Prior: the AALRRSV Model**

Parameter	Min	Max	μ	σ
β	0.9	0.995	0.985	0.005
γ	0.1	100	5	1
ψ	0.1	10	1.5	0.2
η	γ	200	25	5
μ_c	0.012	0.025	0.02	0.001
ρ_x	-0.99	0.99	0.8	0.2
ϕ_x	0.01	0.5	0.15	0.04
λ	1	10	3	0.8
ϕ_d	0.5	10	3	0.8
μ_s	0.001	0.1	0.02	0.005
ρ_s	0.3	0.99	0.8	0.2
σ_w	1e-5	0.001	2.5e-4	6.25e-5

Table 5: **Prior: the EZLRRSV Model**

Parameter	Min	Max	μ	σ
β	0.9	0.995	0.985	0.005
γ	0.1	100	5	1
ψ	0.1	10	1.5	0.2
μ_c	0.012	0.025	0.019	0.001
ρ_x	-0.99	0.99	0.80	0.20
ϕ_x	0.01	0.5	0.15	0.04
λ	1	10	3	0.8
ϕ_d	0.5	10	3	0.8
ϕ_c	1	10	3	0.8
μ_s	0.001	0.10	0.02	0.005
ρ_s	0.30	0.99	0.8	0.2
σ_w	1e-5	0.001	2.5e-4	6.25e-5

4 GSM estimation results with augmented priors

We also perform the GSM Bayesian estimation with augmented priors taking into account moments of asset returns and consumption and dividend growth. The aim of this estimation is to examine whether our GSM estimation results reported in the paper are robust to the augmented priors. The augmented prior on moments is specified to be the product of independent normal density functions as

$$\bar{\xi}\left(\boldsymbol{m}\right) = \prod_{k=1}^{\bar{n}} N\left[m_k | \left(m_k^*, \sigma_{m_k}^2\right)\right]$$

where $\boldsymbol{m} \equiv (m_1, m_2, ..., m_{\bar{n}})$ is a vector of moments under consideration. The location and scale parameters for the moment m_k are m_k^* and σ_{m_k} respectively. We use the following location parameter values for eight moments to form the prior.

$$\mathbb{E}(r_t^f) = 0.014, \quad \sigma(r_t^f) = 0.028, \quad \mathbb{E}(r_t) = 0.068, \quad \sigma(r_t) = 0.18$$
$$\mathbb{E}(\Delta c_t) = 0.018, \quad \sigma(\Delta c_t) = 0.021, \quad \mathbb{E}(\Delta d_t) = 0.018, \quad \sigma(\Delta d_t) = 0.14$$

The scale parameters are set at values such that the prior put 95% of its mass on being within 10% of its location parameter. We simulate these moments from asset pricing models in the GSM Bayesian estimation. The results reported below show that the GSM estimation with the augmented priors yield similar results to those reported in the paper.

Table 6: **GSM Estimation Results: the AAMS Model**

	Prior				Posterior			
Parameter	Mean	Median	5%	95%	Mean	Median	5%	95%
β	0.991	0.991	0.989	0.991	0.980	0.980	0.977	0.983
γ	4.847	4.750	2.750	6.750	2.219	2.281	0.766	3.531
ψ	0.616	0.563	0.563	0.813	2.202	2.180	1.836	2.680
η	13.155	13.500	11.500	13.500	5.557	5.219	4.594	7.922
p_{ll}	0.405	0.406	0.406	0.406	0.860	0.866	0.764	0.923
p_{hh}	0.812	0.813	0.813	0.813	0.997	0.997	0.996	0.997
μ_l	-0.043	-0.043	-0.043	-0.043	-0.048	-0.048	-0.054	-0.041
μ_h	0.033	0.033	0.033	0.033	0.020	0.020	0.018	0.021
λ	2.803	2.750	2.750	3.250	2.791	2.734	2.391	3.547
σ_c	0.006	0.006	0.006	0.006	0.020	0.020	0.018	0.024
σ_d	0.130	0.133	0.117	0.133	0.135	0.136	0.124	0.146
MCMC repetitions	10,000				12,000			

This table presents prior and posterior marginal means, medians, 5 and 95 percentiles of model parameters for the AAMS model. The GSM estimation imposes the augmented prior on moments of asset returns and consumption and dividend growth. MCMC repetitions after transients have dissipated are reported for both the prior and posterior. Estimation results are for annual data 1941–2015.

Table 7: **GSM Estimation Results: the AAMSTV Model**

	Prior				Posterior			
Parameter	Mean	Median	5%	95%	Mean	Median	5%	95%
β	0.989	0.989	0.989	0.989	0.978	0.978	0.974	0.982
γ	4.982	5.250	3.250	6.250	0.848	0.844	0.219	1.531
ψ	0.450	0.438	0.438	0.438	1.779	1.715	1.434	2.199
η	9.496	9.500	9.500	9.500	10.232	9.281	8.531	15.500
p_{ll}^{μ}	0.282	0.281	0.281	0.281	0.706	0.728	0.611	0.774
p_{hh}^{μ}	0.812	0.813	0.813	0.813	0.998	0.999	0.997	0.999
μ_l	-0.066	-0.066	-0.066	-0.066	-0.055	-0.054	-0.060	-0.050
μ_h	0.033	0.033	0.033	0.033	0.018	0.018	0.016	0.019
p_{ll}^{σ}	0.863	0.859	0.734	0.984	0.989	0.989	0.982	0.993
p_{hh}^{σ}	0.840	0.859	0.703	0.953	0.989	0.990	0.984	0.993
σ_l	0.006	0.005	0.005	0.009	0.013	0.013	0.006	0.021
σ_h	0.006	0.006	0.006	0.010	0.029	0.029	0.026	0.032
λ	3.064	3.250	2.750	3.250	2.570	2.547	1.984	3.172
σ_d	0.107	0.102	0.086	0.117	0.134	0.134	0.122	0.146
MCMC repetitions	10,000				12,000			

This table presents prior and posterior marginal means, medians, 5 and 95 percentiles of model parameters for the AAMSTV model. The GSM estimation imposes the augmented prior on moments of asset returns and consumption and dividend growth. MCMC repetitions after transients have dissipated are reported for both the prior and posterior. Estimation results are for annual data 1941–2015.

Table 8: **GSM Estimation Results: the AALRRSV Model**

	Prior				Posterior			
Parameter	Mean	Median	5%	95%	Mean	Median	5%	95%
β	0.989	0.989	0.985	0.993	0.992	0.992	0.990	0.995
γ	8.967	8.875	7.125	10.875	8.013	8.031	7.156	8.906
ψ	1.155	1.156	0.906	1.406	0.899	0.895	0.855	0.957
η	33.125	33.000	25.000	41.000	31.085	31.500	19.500	41.500
μ_c	0.019	0.019	0.018	0.020	0.020	0.020	0.019	0.022
ρ_x	0.844	0.844	0.844	0.844	0.940	0.940	0.927	0.953
ϕ_x	0.311	0.305	0.273	0.367	0.209	0.210	0.181	0.233
λ	3.239	3.125	2.875	3.875	2.784	2.734	2.453	3.203
ϕ_d	4.900	4.875	4.375	5.375	4.995	4.984	4.609	5.359
μ_s	0.011	0.010	0.010	0.011	0.020	0.020	0.019	0.020
ρ_s	0.969	0.969	0.969	0.969	0.950	0.950	0.950	0.950
σ_w	1.84E-04	1.68E-04	1.68E-04	1.98E-04	2.08E-04	2.09E-04	1.93E-04	2.18E-04
MCMC repetitions	10,000				12,000			

This table presents prior and posterior marginal means, medians, 5 and 95 percentiles of model parameters for the AALRRSV model. The GSM estimation imposes the augmented prior on moments of asset returns and consumption and dividend growth. MCMC repetitions after transients have dissipated are reported for both the prior and posterior. Estimation results are for annual data 1941–2015.

Table 9: **GSM Estimation Results: the EZMS Model**

	Prior				Posterior			
Parameter	Mean	Median	5%	95%	Mean	Median	5%	95%
β	0.986	0.985	0.985	0.987	0.981	0.981	0.978	0.984
γ	10.221	10.250	10.250	10.250	4.013	3.953	3.469	4.672
ψ	0.421	0.438	0.313	0.438	2.397	2.359	1.859	2.984
p_{ll}	0.343	0.344	0.344	0.344	0.904	0.897	0.860	0.951
p_{hh}	0.812	0.813	0.813	0.813	0.996	0.997	0.992	0.997
μ_l	-0.059	-0.059	-0.059	-0.059	-0.039	-0.041	-0.052	-0.018
μ_h	0.029	0.029	0.029	0.029	0.022	0.021	0.018	0.029
λ	3.252	3.250	3.250	3.250	3.192	3.172	2.641	3.766
σ_c	0.006	0.006	0.006	0.006	0.019	0.019	0.016	0.023
σ_d	0.105	0.102	0.102	0.117	0.132	0.133	0.117	0.145
MCMC repetitions	10,000				12,000			

This table presents prior and posterior marginal means, medians, 5 and 95 percentiles of model parameters for the EZMS model. The GSM estimation imposes the augmented prior on moments of asset returns and consumption and dividend growth. MCMC repetitions after transients have dissipated are reported for both the prior and posterior. Estimation results are for annual data 1941–2015.

Table 10: **GSM Estimation Results: the EZMSTV Model**

	Prior				Posterior			
Parameter	Mean	Median	5%	95%	Mean	Median	5%	95%
β	0.991	0.991	0.991	0.991	0.989	0.989	0.986	0.992
γ	7.995	8.250	6.750	8.250	7.537	7.406	6.656	8.594
ψ	0.975	0.938	0.938	1.063	1.168	1.148	1.039	1.383
p_{ll}^{μ}	0.595	0.594	0.594	0.594	0.609	0.610	0.579	0.632
p_{hh}^{μ}	0.812	0.813	0.813	0.813	0.975	0.975	0.971	0.978
μ_l	-0.035	-0.035	-0.035	-0.035	-0.060	-0.060	-0.066	-0.056
μ_h	0.037	0.037	0.037	0.037	0.022	0.022	0.020	0.024
p_{ll}^{σ}	0.856	0.859	0.766	0.984	0.992	0.994	0.979	0.998
p_{hh}^{σ}	0.837	0.828	0.734	0.953	0.923	0.921	0.866	0.979
σ_l	0.006	0.005	0.005	0.009	0.014	0.014	0.010	0.016
σ_h	0.006	0.006	0.006	0.006	0.030	0.029	0.023	0.038
λ	3.271	3.250	3.250	3.750	2.205	2.242	1.711	2.836
σ_d	0.102	0.102	0.086	0.117	0.132	0.132	0.116	0.151
MCMC repetitions	10,000				12,000			

This table presents prior and posterior marginal means, medians, 5 and 95 percentiles of model parameters for the EZMSTV model. The GSM estimation imposes the augmented prior on moments of asset returns and consumption and dividend growth. MCMC repetitions after transients have dissipated are reported for both the prior and posterior. Estimation results are for annual data 1941–2015.

Table 11: **GSM Estimation Results: the EZLRRSV Model**

	Prior				Posterior			
Parameter	Mean	Median	5%	95%	Mean	Median	5%	95%
β	0.993	0.993	0.989	0.993	0.984	0.984	0.981	0.986
γ	6.800	6.750	5.250	7.750	8.542	8.438	6.938	10.438
ψ	1.375	1.313	1.313	1.688	2.773	2.789	2.531	3.023
μ_c	0.020	0.020	0.019	0.021	0.022	0.023	0.021	0.024
ρ_x	0.937	0.938	0.938	0.938	0.985	0.989	0.971	0.990
ϕ_x	0.290	0.305	0.195	0.336	0.077	0.071	0.060	0.108
λ	2.582	2.250	2.250	3.750	4.518	4.469	3.688	5.406
ϕ_d	4.742	4.750	3.750	5.750	4.546	4.531	4.156	4.938
ϕ_c	4.002	4.250	2.750	5.250	1.217	1.156	1.016	1.656
μ_s	0.013	0.011	0.011	0.017	0.019	0.019	0.019	0.020
ρ_s	0.938	0.938	0.938	0.938	0.950	0.950	0.950	0.950
σ_w	1.50E-04	1.37E-04	1.37E-04	1.68E-04	1.93E-04	1.93E-04	1.87E-04	2.03E-04
MCMC repetitions	10,000				12,000			

This table presents prior and posterior marginal means, medians, 5 and 95 percentiles of model parameters for the EZLRRSV model. The GSM estimation imposes the augmented prior on moments of asset returns and consumption and dividend growth. MCMC repetitions after transients have dissipated are reported for both the prior and posterior. Estimation results are for annual data 1941–2015.

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