Cash Flows Discounted Using a Model Free SDF Extracted Under a Yield Curve Prior

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Paper: http://www.aronaldg.org/papers/dcf.pdf
Slides: http://www.aronaldg.org/papers/dcfclr.pdf

Contribution

- Introduce a model-free, Bayesian extraction procedure for the stochastic discount factor (SDF) under a yield curve prior.
  - The prior enforces external information that U.S. short- and long-term real interest rates are low.
  - No theory of the SDF is used in the extraction other than moment restrictions implied by its definition.
  - Previous methods directly or indirectly use an asset pricing model; e.g., a factor representation or long run risks.
- Use the extracted SDF to determine the cash flow risk premia on a panel of industrial profits and consumption.
  - Computational accuracy compels the use of Gaussian autoregressions to compute risk premia over long horizons rather than simulation from a nonparametric model.

Findings

- The risk premia on industrial cash flows show a decreasing term structure for 1 to 50 year horizons.
  - The exception is retail trade which is a hedging asset; more so in the short run than the long run.

- The risk premia on the consumption cash flow are low in the short term but increase to 4 percent per year 50 years out.

- The extracted term structures of equity risk premia generally confirm the limited information (bounded rationality) model of Croce, Lettau, and Ludvigson (2015, *RFS*).

Stochastic Discount Factor

- The stochastic discount factor $SDF_{0,t}$ discounts a future payoff CF_t to its present value $PV_{0,t}$ given the information $\mathcal{F}_0$ available at time $t = 0$.

- One step ahead

$$PV_{0,1} = \mathcal{E}\left(SDF_{0,1}\, CF_1 \,|\, \mathcal{F}_0\right)$$

- More than one step ahead

$$PV_{0,t} = \mathcal{E}\left(SDF_{0,t}\, CF_t \,|\, \mathcal{F}_0\right) = \mathcal{E}\left(\prod_{s=1}^{t} SDF_{s-1,s}\, CF_t \,|\, \mathcal{F}_0\right)$$

- Yield (geometric) on an t-year, risk-free bond

$$Y_{0,t} = -\log \mathcal{E}\left(SDF_{0,t} \,|\, \mathcal{F}_0\right)/t = -\log \mathcal{E}\left(\prod_{s=1}^{t} SDF_{s-1,s} \,|\, \mathcal{F}_0\right)/t$$

Stochastic Discount Factor Extraction

Parameter:

$$\begin{aligned}\theta &= (SDF_{1929,1930}, \ldots, SDF_{t-1,t}, \ldots, SDF_{2014,2015}) \\ &= (\theta_1, \ldots, \theta_t, \ldots, \theta_{86})\end{aligned}$$

Data: A 28 by $n = 86$ matrix x with columns

25 Fama-French portfolios, $R_{st}, s = 1, \ldots, 25$, real, gross
30 day T-bill returns, R_{bt}, real, gross
Consumption growth, $\frac{C_t}{C_{t-1}}$, real, per-capita
Labor income growth, $\frac{L_t}{L_{t-1}}$, real, per-capita

Prior: $Y_{t,t+s}$ = yield on an s-year bond at time t implied by θ

$$\pi(\theta) = \prod_{t=1}^{n} \phi[(Y_{t,t+1} - 0.00896)/0.01]\phi[(Y_{t,t+30} - 0.02)/0.01)]$$

Assumptions for Extraction

- Two implications of the definition of the SDF

$$0 = \mathcal{E}\left[m(x_t, x_{t-1}, \theta_t)\right] = \mathcal{E}\left\{\begin{pmatrix} R_{s,t-1} - 1 \\ R_{b,t-1} - 1 \\ C_{t-1}/C_{t-2} - 1 \\ L_{t-1}/L_{t-2} - 1 \\ 1 \end{pmatrix} \otimes \left[1 - \theta_t \begin{pmatrix} R_{s,t} \\ R_{b,t} \end{pmatrix}\right]\right\}$$

$$0 = \mathcal{E}\left[m(x_t, x_{t-1}, \theta_t)\right]' \left[m(x_s, x_{s-1}, \theta_s)\right] \quad t \neq s$$

- A factor error structure

$$\text{Var}\left[(U_v \otimes U_e)' m(x_t, x_{t-1}, \theta)\right] = \text{a diagonal matrix}$$

where U_v and U_e are comprised of orthonormal eigenvectors that do not depend on θ or t.

- Positivity: $\theta_i > 0, \; i = 1, \ldots, 86$

Conceptual Issues

- Bayesian inference requires a likelihood $p(x \mid \theta)$

- We are unwilling to assume more than just stated, in particular, unwilling to assume a specific general equilibrium model.

- x and θ are endogenous so that determining a likelihood $p(x \mid \theta)$ requires some care

Resolution of Conceptual Issues - References

Gallant, A. Ronald, and Han Hong (2007), "A Statistical Inquiry into the Plausibility of Recursive Utility," *Journal of Financial Econometrics* 5, 523–559.

Gallant, A. Ronald (2016), "Reflections on the Probability Space Induced by Moment Conditions with Implications for Bayesian Inference," *Journal of Financial Econometrics* 14, 284–294.

Gallant, A. Ronald (2016), "Reply to Comment on Reflections," *Journal of Financial Econometrics* 14, 284–294.

Gallant, A. Ronald (2018), "Complementary Bayesian Method of Moments Strategies," http://www.aronaldg.org/papers/cb.pdf.

Resolution of Conceptual Issues - Z

Base inference on

$$Z(x,\theta) = \sqrt{n}\, S_n^{-1/2}(\theta)(U_z \otimes U_e)'\bar{m}_n(x,\theta)$$

- $\bar{m}(x,\theta) = \frac{1}{n}\sum_{t=2}^{n} m(x_t, x_{t-1}, \theta_t)$

- $S_n(\theta)$ a diagonal matrix with elements

$$s_i(\theta) = \frac{1}{n}\sum_{t=2}^{n}\left(h_{t,i}(\theta) - \frac{1}{n}\sum_{t=2}^{n} h_{t,i}(\theta)\right)^2 .$$

- $h_{t,i}(\theta)$ the elements of

$$H_t(\theta) = (U_v \otimes U_e)' m(x_t, x_{t-1}, \theta)$$

Conceptual Issues - GE P-Space

- Assume a general equilibrium model exists: Implies a marginal density $p^o(x, \theta)$ and hence a conditional $p^o(x \,|\, \theta)$ exist.

- Were $p^o(x \,|\, \theta)$ known the relevant P-measure for Bayesian inference is P^o on $\mathcal{X} \times \Theta$ with density $p^o(x \,|\, \theta)\pi(\theta)$ where $\pi(\theta)$ is the prior.
  - If practicable, must calibrate some nuisance parameters.
  - We assume nuisance parameters have been calibrated by Nature because their values are not needed.

- The GE probability space is, therefore,

$$(\mathcal{X} \times \Theta, \mathcal{C}^o, P^o),$$

where $\mathcal{C}^o$ denotes the Borel subsets of $\mathcal{X} \times \Theta$.

Conceptual Issues - Moment Induced P-Space

- The random variable $z = Z(x, \theta)$ over $(\mathcal{X} \times \Theta, \mathcal{C}^o, P^o)$ has a distribution Ψ with a support $\mathcal{Z}$.

- Let $\mathcal{C}$ be the smallest σ-algebra containing the preimages $C = Z^{-1}(B)$ where B ranges over the Borel subsets of $\mathcal{Z}$.

- The probability of $C \in \mathcal{C}$ is $P[C = Z^{-1}(B)] = \int_B d\Psi(z)$

- The moment induced probability space is, therefore,

$$(\mathcal{X} \times \Theta, \mathcal{C}, P)$$

Conceptual Issues - Extension of MI P-Space

- Define $\mathcal{C}^*$ to be the smallest σ-algebra that contains all sets in $\mathcal{C}$ plus all sets of the form $R_B = (\mathcal{X} \times B)$, where B is a Borel subset of Θ.

- If $Z(x, \theta)$ is a semi-pivotal, there is an extension to a space

$$(\mathcal{X} \times \Theta, \mathcal{C}^*, P^*)$$

  such that $P^o(C) = P^*(C)$ for all $C \in \mathcal{C}^*$. (Gallant, 2016)

  - Sufficient to be semi-pivotal is that Z is continuous and unbounded in at least one element of x

- The σ-algebras involved satisfy $\mathcal{C} \subset \mathcal{C}^* \subset \mathcal{C}^o$.

Conceptual Issues - MI Likelihood

- The "method of moments representation" of the likelihood on the extended space is $(\mathcal{X} \times \Theta, \mathcal{C}^*, P^*)$ is

$$p^*(x \mid \theta) = \text{adj}(x, \theta)\psi[Z(x, \theta)]$$

  where $\text{adj}(x, \theta)$ is analogous to a Jacobian term.

  – Negligible for small ($\leq$ 30) samples in most applications.

  – Omission equivalent using a data dependent prior.

- The key insight that allows substitution of the "method of moments representation" of the likelihood for the likelihood under the GE model in a Bayesian analysis is the fact that both probability measures P^o and P^* assign the same probability to sets in $\mathcal{C}^*$.

Conceptual Issues - Information Loss

- Because $\mathcal{C}^*$ is a subset of $\mathcal{C}^o$, some information is lost

- Intuitively this is similar to the information loss that occurs when one divides the range of a continuous variable into intervals and uses a discrete distribution to assign probability to each interval. Both the continuous and discrete distributions assign the same probability to each interval but the discrete distribution cannot assign probability to subintervals.

- How much information is lost depends on how well one chooses moment conditions.

Summary

- Moment conditions:

$$Z(x,\theta) = \sqrt{n}\, S_n^{-1/2}(\theta)(U_z \otimes U_e)'\bar{m}_n(x,\theta)$$

  – 754 moment equations, 86 observations, 86 parameters

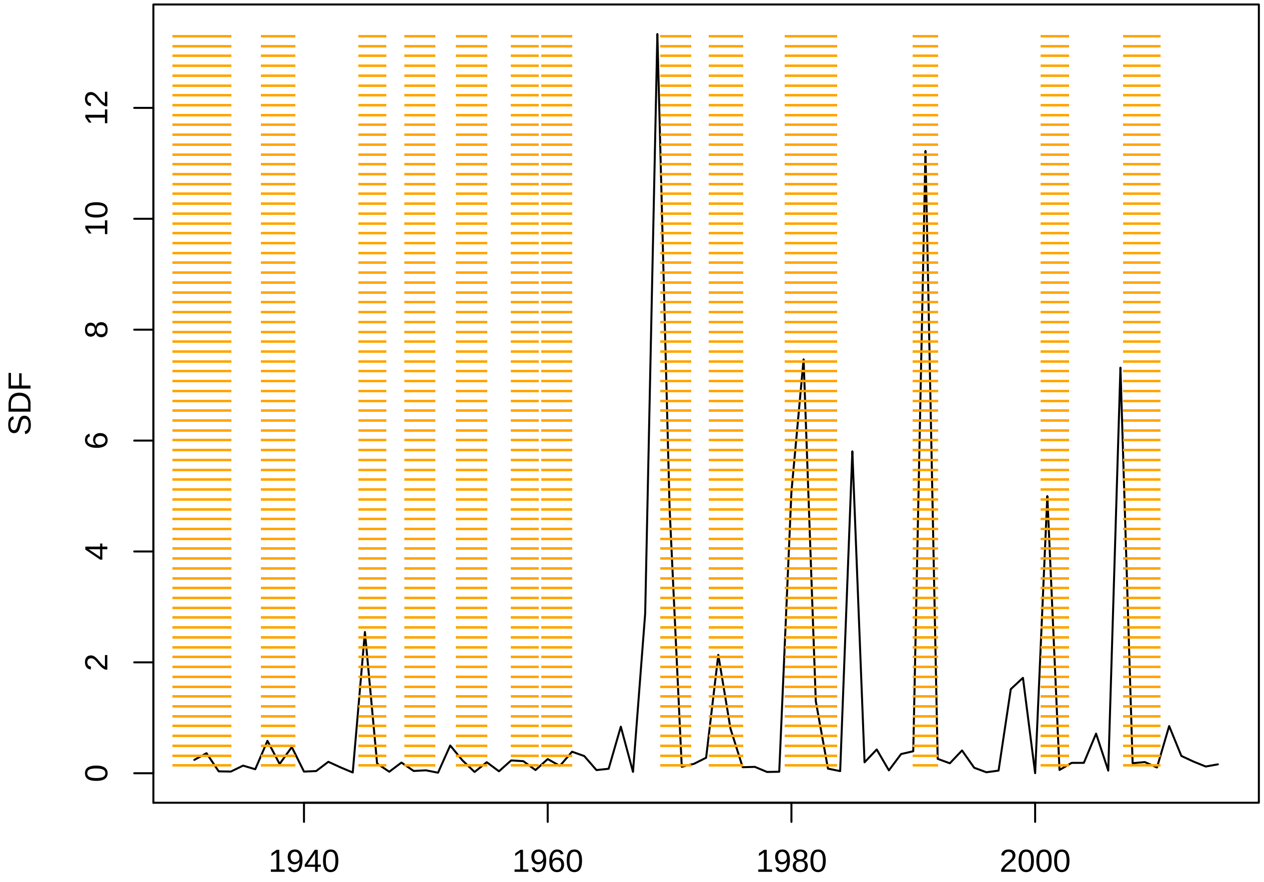
- Prior

$$\pi(\theta) = \prod_{t=1}^{n} \phi[(Y_{1,t} - 0.00896)/0.01]\phi[(Y_{30,t} - 0.02)/0.01)]$$

- Use normal density $\phi(z)$ as $\psi(z)$, omit adjustment term

- Estimation by Markov chain Monte Carlo (MCMC)

Figure 1. The Posterior Mode of the SDF



Discounted Cash Flow Computation

- Consider

$$y_t = \begin{pmatrix} \log(SDF_{t-1,t}) \\ \log(GDP_t) - \log(CF_t) \\ \log(CF_t) - \log(CF_{t-1}) \end{pmatrix}$$

- $SDF_{t-1,t}$ is the extracted Stochastic Discount Factor
  CF_t a real, per-capita, payoff at time t, e.g., corporate profits
  GDP_t real, per-capita, Gross Domestic Product at time t

- Fit $y_t \sim N(b + By_{t-1}, \Sigma)$ subject to the yield curve prior

- From $(\hat{b}, \hat{B}, \hat{\Sigma})$, present value $PV_{0,t}(CF)$ of CF_t, expected value $EV_{0,t}(CF)$, and yield $Y_{0,t} = -\log[PV_{0,t}(1))]/t$ can be computed analytically using tedious formulae that are in the paper.

Addendum Regarding the Prior - 1

Yields for the SDF extraction prior

$$\pi(\theta) = \prod_{t=1}^{n} \phi[(Y_{t,t+1} - 0.00896)/0.01]\phi[(Y_{t,t+30} - 0.02)/0.01)]$$

were computed as in the previous slide with

$$y_t = \begin{pmatrix} \log(\theta_t) \\ \log(GDP_t) - \log(GDP_{t-1}) \end{pmatrix}$$

Footnote: $\log(GDP_t) - \log(GDP_{t-1})$ is in the information set of y_t of the previous slide

Addendum Regarding the Prior - 2

- Campbell (2003, p. 812): the average short-term U.S. real rate was 0.896 percent over the period 1947–1998

- Tesar and Obstfeld (2015) Figure 2: the 10-year real rate of interest over 1930–2014 was often negative, generally fluctuated between 0 and 2.5 percent, and only briefly bumped 5 percent prior to WWII and in the early 1980s.

Addendum Regarding the Prior - 3

Table 1. TIPS (real) Yields

Year	5-year	7-year	10-year	20-year
2004	1.02	1.39	1.76	2.13
2005	1.50	1.63	1.81	1.97
2006	2.28	2.30	2.31	2.31
2007	2.15	2.25	2.29	2.36
2008	1.30	1.63	1.77	2.18
2009	1.06	1.32	1.66	2.21
2010	0.26	0.68	1.15	1.73
2011	-0.41	0.10	0.55	1.20
2012	-1.20	-0.88	-0.48	0.21
2013	-0.76	-0.30	0.07	0.75
2014	-0.09	0.32	0.44	0.86
2015	0.15	0.36	0.45	0.78

Risk Premium on a Cash Flow

- Risk free (continuously compounded) rate

$$\left(e^{t\,r^f_{0,t}}\right) PV_{0,t}(1) = 1 \quad \text{or} \quad r^f_{0,t} = -\log(PV_{0,t})/t$$

  – Derivation: $dP_t/dt = rP_t \implies P_t = P_0 e^{tr}$; initial condition $P_t = 1 \implies P_0 = e^{-tr} \implies r = -\log(P_0)/t$

- Extension to a cash flow (cash flow yield curve)

$$\left(e^{t\,r_{0,t}}\right) PV_{0,t}(CF) = EV_{0,t}(CF) \quad \text{or} \quad r_{0,t} = -\log\left(\frac{PV_{0,t}}{EV_{0,t}}\right)/t$$

- Risk premium on cash flow CF_t

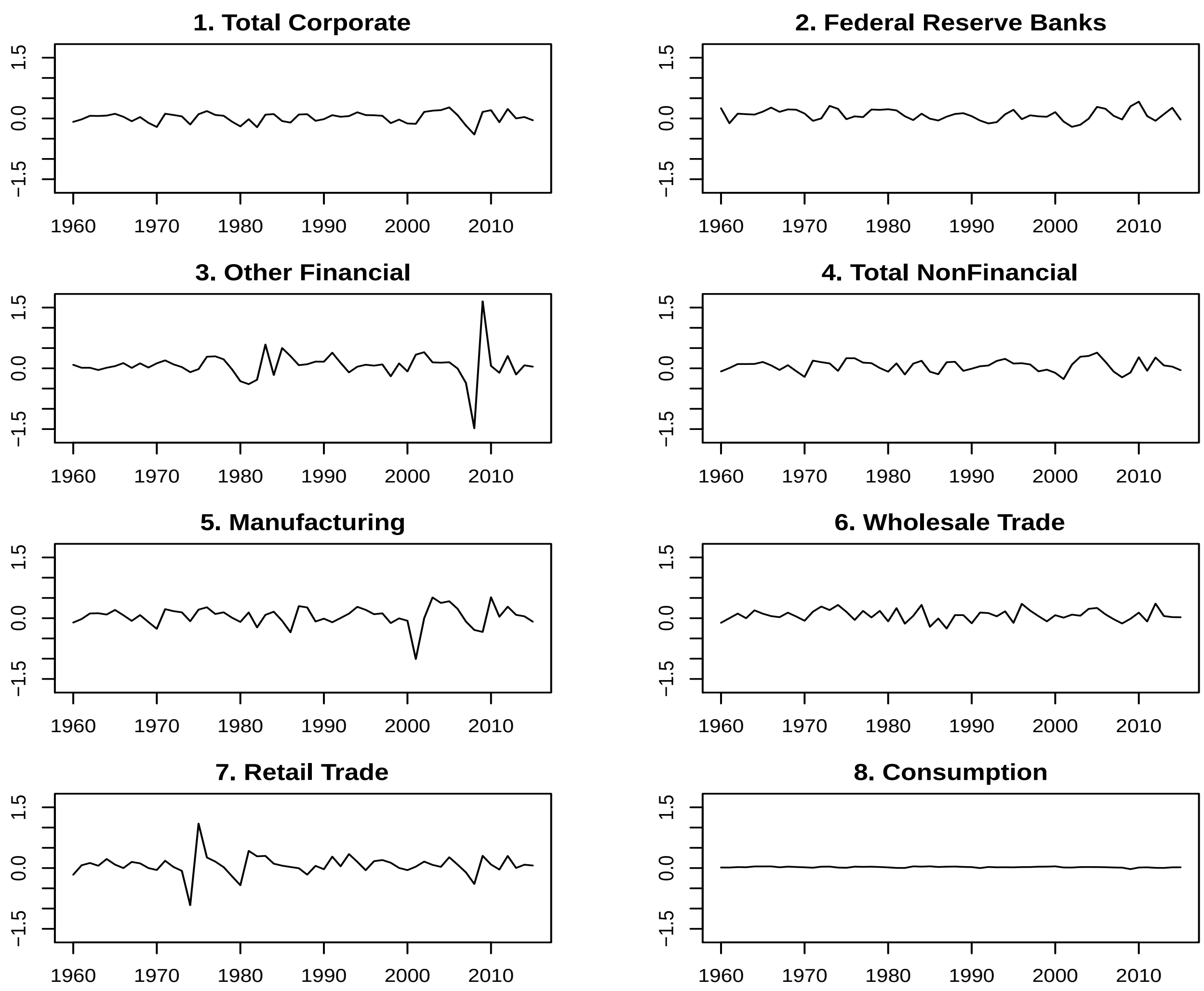
$$r_{0,t} - r^f_{0,t}$$

Cash Flow Data

Table 2. Summary Statistics

	Industry	$gdp_t - cf_t$		$\Delta cf_{t-1,t}$	
		Mean	Std Dev	Mean	Std Dev
1	Total Corporate	2.572	0.227	0.019	0.131
2	Federal Reserve Banks	5.834	0.343	0.045	0.131
3	Other Financial	4.366	0.468	0.028	0.360
4	Total NonFinancial	2.828	0.272	0.016	0.146
5	Manufacturing	3.626	0.473	0.004	0.239
6	Wholesale Trade	5.194	0.260	0.027	0.137
7	Retail Trade	5.141	0.332	0.028	0.252
8	Consumption (NDS)	0.597	0.063	0.022	0.013
		$sdf_{t-1,t}$		$gdp_t - gdp_{t-1}$	
	log-MRS and GDP growth	-1.2231	1.8595	0.0196	0.0202

Figure 2. Real Cash Flow Growth



Cash Flow Yield Curves

Table 3. Risk Premia on Stripped Cash Flows

Industry	Exposure		Average Risk Premium, Horizons in Years					
	Cov	Corr	1	1–10	11–20	21–30	31–40	41–50
1 Total Corporate	-0.083	-0.37	0.083	0.104	0.079	0.059	0.049	0.042
2 Fed Reserve Banks	-0.042	-0.21	0.042	0.107	0.099	0.073	0.056	0.045
3 Other Financial	-0.138	-0.23	0.138	0.174	0.115	0.077	0.057	0.045
4 Total NonFinancial	-0.087	-0.36	0.087	0.102	0.086	0.071	0.060	0.054
5 Manufacturing	-0.161	-0.41	0.161	0.158	0.129	0.108	0.094	0.085
6 Wholesale Trade	-0.082	-0.34	0.082	0.104	0.088	0.070	0.056	0.046
7 Retail Trade	0.044	0.11	-0.044	-0.070	-0.053	-0.035	-0.023	-0.014
8 Consumption	-0.010	-0.35	0.010	0.022	0.033	0.037	0.039	0.040

Findings

- The risk premia on industrial cash flows show a decreasing term structure for 1 to 50 year horizons.
  - The exception is retail trade which is a hedging asset; more so in the short run than the long run.

- The risk premia on the consumption cash flow are low in the short term but increase to 4 percent per year 50 years out.

- The extracted term structures of equity risk premia generally confirm the limited information (bounded rationality) model of Croce, Lettau, and Ludvigson (2015, *RFS*).