Online Appendix for Variance-Covariance from a Metropolis Chain on a Curved, Singular Manifold*

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`Code: www.aronaldg.org/webfiles/npb`

`Paper: www.aronaldg.org/papers/sdev.pdf`

`Slides: www.aronaldg.org/papers/npbsdev.pdf`

`Appendix: www.aronaldg.org/papers/sdev_appendix.pdf`

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Abstract

We consider estimation of variance and covariance from a point cloud that are draws from a posterior distribution that lie on a curved, singular manifold. The motivating application is Bayesian inference regarding a likelihood subject to overidentified moment equations using MCMC (Markov Chain Monte Carlo). The MCMC draws lie on a singular manifold that typically is curved. Variance and covariance are Euclidean concepts. A curved, singular manifold is not typically a Euclidean space. We explore some suggestions on how to adapt a Euclidean concept to a non-Euclidean space then build on them to propose and illustrate appropriate methods.

Keywords and Phrases: Method of moments, Bayesian inference, Simultaneously valid credibility intervals, Point cloud, Curved, singular manifold.

JEL Classification: C11, C14, C15, C32, C36, C58

Tables and Figures

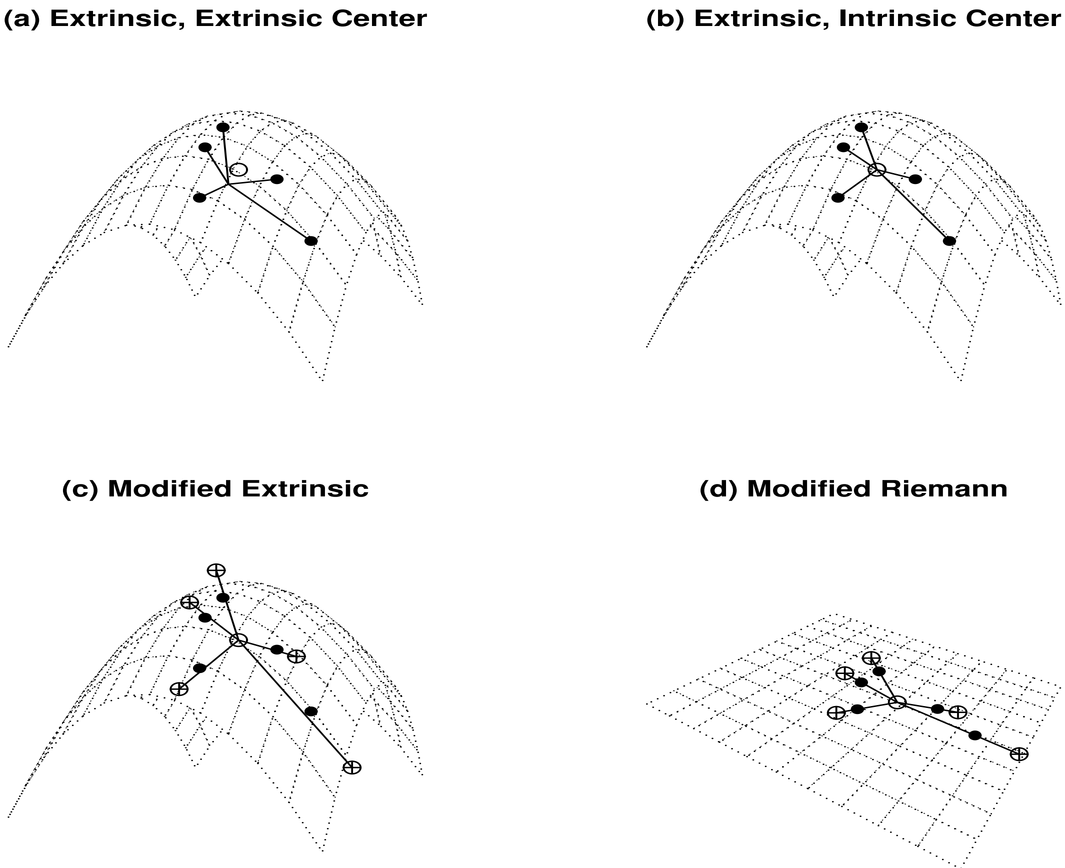

Figure 1. Illustration of Scale Measures Panels (a), (b), and (c) show a hypothetical surface with hypothetical sample points shown as solid dots, ●, and the intrinsic mean, $\bar{x}$, shown as an open circle, ○. Panel (d) shows the plane tangent to the hypothetical surface at the intrinsic mean with the solid dots and open circle projected onto that plane.

In Panel (a) are vectors formed by connecting the extrinsic mean, $\tilde{x}$, to the sample points, ●. The scale measure V_{EC} is the average of the outer product of these vectors. This is the standard measure of scale, S^2, for any sample.

In Panel (b) are vectors formed by connecting the intrinsic mean, $\bar{x}$, to the sample points, The scale measure V_{IC} is the average of the outer product of these vectors.

In Panel (c) are vectors formed by extending the vectors of Panel (b) by the length of their geodesics, coordinate by coordinate, to connect to the points shown as circled pluses, $\oplus$. Because the multiples of coordinates can differ, the circled plus vectors need not pass through the sample points. The scale measure V_{ME} is the average of the outer product of the circled plus vectors.

In Panel (d) are vectors on the tangent plane $T_{\bar{x}}M$ that are formed by extending the vectors connecting the intrinsic mean, ○. to the projected sample points, ● to the points shown as circled pluses, $\oplus$, by the length of the geodesics connecting ○ to ● on the manifold M. The circled plus vectors will pass through the projected sample points. The scale measure V_{MR} is the average of the outer product of the circled plus vectors.

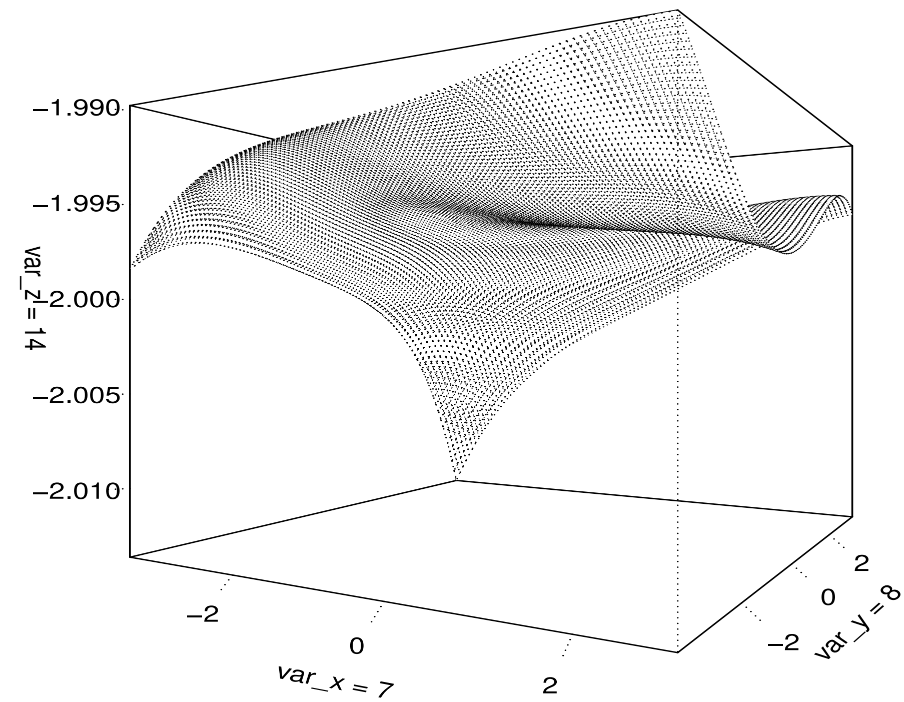

Figure 2. 95% Credibility Region, Demand and Supply Example. The demand and supply example is described in Subsection 4.1. The coordinates `var_x` and `var_y` are the seventh and eighth chart coordinates, that is, they are the coefficients of the seventh and eighth columns of $T_{\bar{\mathsf{x}}}$; `var_z` is the last element of $\mathsf{x} = (\rho, \theta)$. It is the price elasticity of demand. All other chart coordinates are held fixed at the values of the intrinsic mean. The surface was obtained by fitting a multivariate polynomial of degree four with draws $\{\mathsf{x}_i\}$ as the dependent variable and the corresponding points $\{z_i\}$ on the chart as the independent variable. In this instance, `var_x` is roughly interpretable as ρ_{10}, which is $P_{1,1}$, and `var_y` is roughly interpretable as ρ_{11}, which is $P_{2,2}$. $P_{1,1}$ and $P_{2,2}$ are the parameters that determine the stochastic volatility of log price and log quantity, respectively.

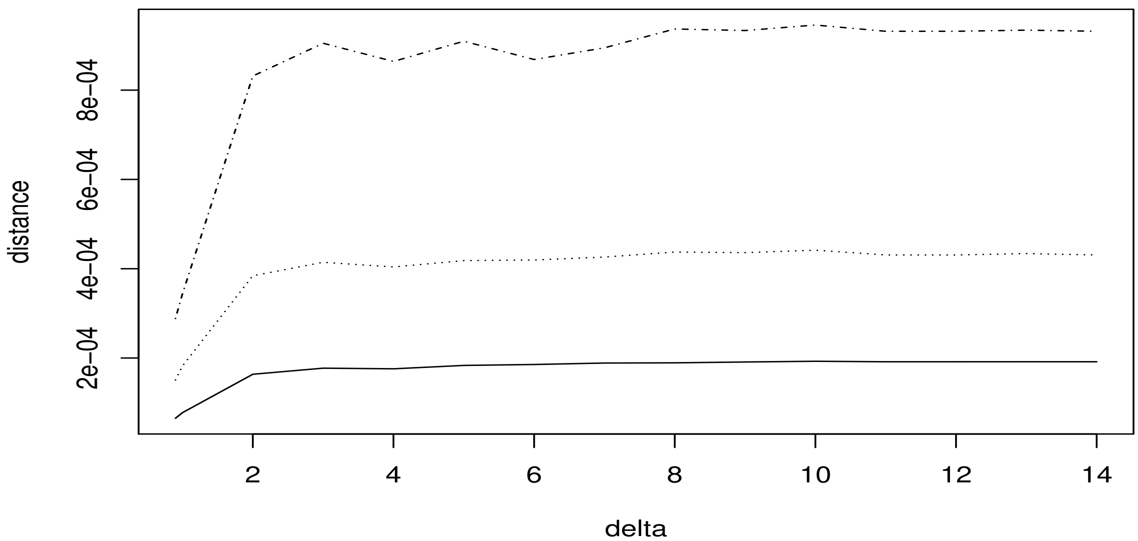

Figure 3. Distance of Edge Midpoints from Manifold, Demand and Supply Example. For the graph $\mathcal{G}_\epsilon$ with offset Δ as shown on the horizontal axis, the distance of the center of each edge from the manifold M is computed. The dotdash line is the 99th percentile, the dotted line is the 90th percentile, and the solid line is the mean.

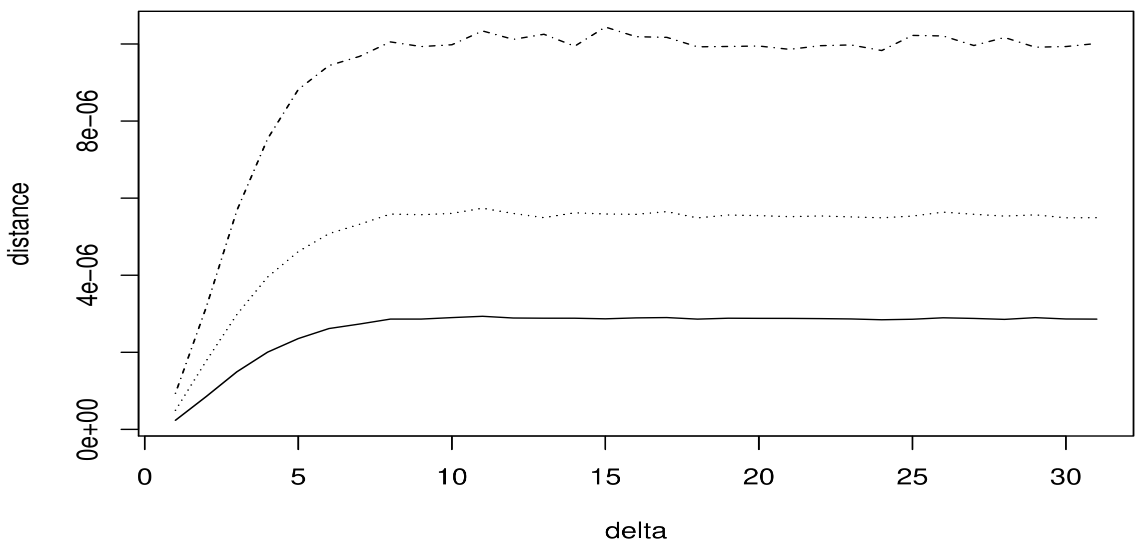

Figure 4. Distance of Edge Midpoints from Manifold, Stochastic Discount Function Example For the graph $\mathcal{G}_\epsilon$ with offset Δ as shown on the horizontal axis, the distance of the center of each edge from the manifold M is computed. The dotdash line is the 99th percentile, the dotted line is the 90th percentile, and the solid line is the mean.

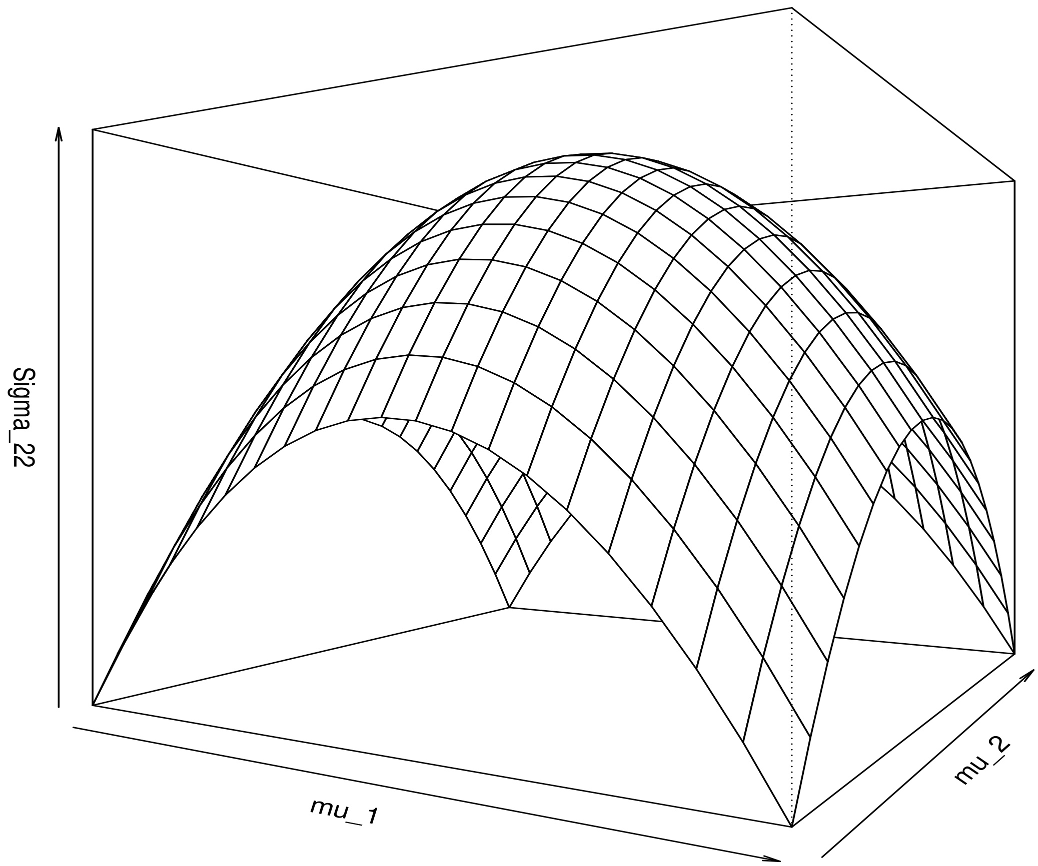

Figure 5. Curved Manifold Example Plotted is the manifold M for the likelihood (39) subject to moment conditions (2) determined by (41) and (42). The missing dimensions, $\Sigma_{1,1}$, $\Sigma_{1,2}$, and θ, are held constant at 5, 6.12372, and 5, respectively.

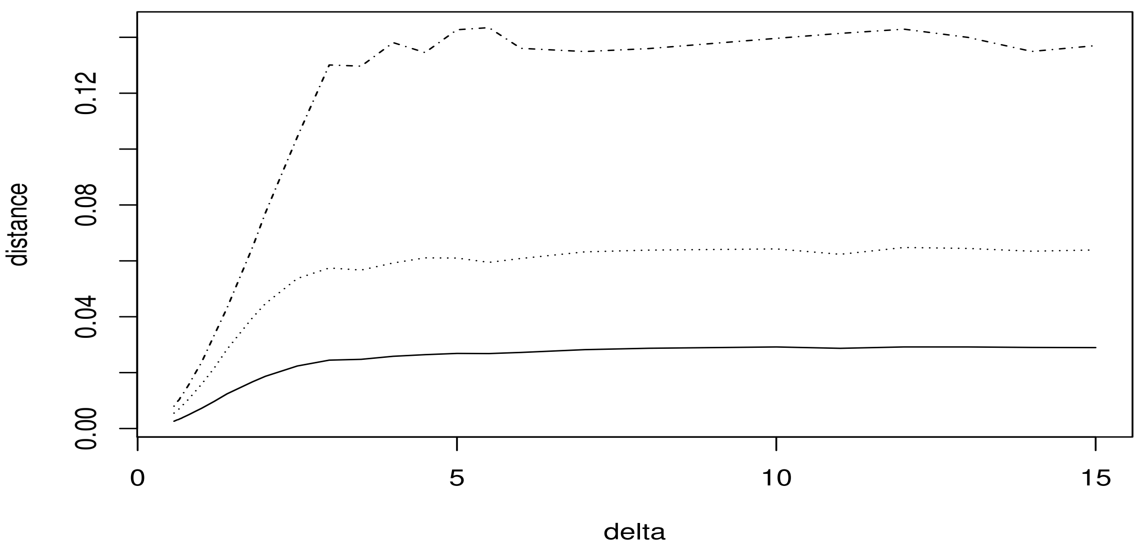

Figure 6. Distance of Edge Midpoints from Manifold, Curved Manifold Example For the graph $\mathcal{G}_\epsilon$ with offset Δ as shown on the horizontal axis, the distance of the center of each edge from the manifold M is computed. The dotdash line is the 99th percentile, the dotted line is the 90th percentile, and the solid line is the mean.

Table 1. Illustration of Population Variances and Correlations

	V_{EC}			C_{EC}	
0.073658	0.058105	0.000088	1.000000	0.788701	0.002893
0.058105	0.073686	0.000041	0.788701	1.000000	0.001337
0.000088	0.000041	0.012574	0.002893	0.001337	1.000000
	V_{IC}			C_{IC}	
0.073658	0.058105	0.000083	1.000000	0.788701	0.002194
0.058105	0.073686	0.000034	0.788701	1.000000	0.000896
0.000083	0.000034	0.019536	0.002194	0.000896	1.000000
	V_{ME}			C_{ME}	
0.073658	0.058105	0.000093	1.000000	0.788701	0.002262
0.058105	0.073686	0.000037	0.788701	1.000000	0.000913
0.000093	0.000037	0.022778	0.002262	0.000913	1.000000
	V_{MR}			C_{MR}	
0.073658	0.058105	0.000000	1.000000	0.788701	0.000000
0.058105	0.073686	0.000000	0.788701	1.000000	0.000000
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

Shown are the variance matrices V_{EC}, V_{IC}, V_{ME}, and V_{MR} and correlation matrices C_{EC}, C_{IC}, C_{ME}, and C_{MR} computed from a simulation of (16) of length $n = 198373$.

Table 2. Demand and Supply Example, $\Delta = 0.9$

	Mean		Standard Deviation			
			Extrinsic		Modified	
Parameter	Extrinsic	Intrinsic	Extr Ctr	Intr Ctr	Extrinsic	Riemann
μ_1	0.006974	-0.000014	0.032410	0.033155	0.056276	0.000416
μ_2	-0.006384	-0.007252	0.034833	0.034843	0.078493	0.055205
μ_3	-0.001982	0.007796	0.035280	0.036610	0.081187	0.057796
$R_{1,1}$	0.995638	0.985301	0.030594	0.032293	0.072945	0.051602
$R_{1,2}$	-0.000188	-0.009587	0.019377	0.021537	0.061537	0.033760
$R_{2,2}$	1.001946	1.050761	0.031913	0.058325	0.093115	0.090660
$R_{1,3}$	-0.004291	-0.003850	0.018963	0.018969	0.054806	0.030062
$R_{2,3}$	0.001238	-0.006318	0.018397	0.019889	0.050946	0.031336
$R_{3,3}$	0.996106	0.984606	0.030197	0.032314	0.076838	0.051241
$P_{1,1}$	0.137964	0.173574	0.081201	0.088667	0.113114	0.142283
$P_{2,2}$	0.004711	-0.012474	0.109165	0.110509	0.123758	0.177366
$P_{3,3}$	-0.058212	-0.058302	0.128347	0.128347	0.138672	0.203544
a_1	11.986857	11.982728	0.010649	0.011422	0.028206	0.018471
a_2	-1.996886	-1.994403	0.006776	0.007217	0.018729	0.011906

The data are a simulation of the demand and supply system (24) through (26). An MCMC chain of length 50,000 was computed using the Surface Sampling Algorithm for the likelihood (27) subject to moment conditions (2) as determined by (29) through (31) The prior for ρ is independent normal with location the unconstrained maximum likelihood estimates of (27) and scale twice the maximum likelihood standard errors. The prior for $\theta = (a_1, a_2)$ is independent normal with means $(12, -2)$ and standard deviations $(2, 2)$. The support conditions on R and P of (28) are that diagonals of R must be positive, the first diagonal element P must be positive, and the eigenvalues of the companion matrix of Σ must be less than one in absolute value. In addition, a_1 must be positive and a_2 negative. The chain was reduced (downsampled) with a stride of 10 leaving a chain of length 5,000 for computations. Means and standard deviations shown in the table for offset $\Delta = 0.9$, which is the smallest value of Δ for which the manifold M_ϵ is connected.

Table 3. Demand and Supply Example, $\Delta = 3$

	Mean		Standard Deviation			
			Extrinsic		Modified	
Parameter	Extrinsic	Intrinsic	Extr Ctr	Intr Ctr	Extrinsic	Riemann
μ_1	0.006974	-0.000005	0.032410	0.033154	0.034008	0.000024
μ_2	-0.006384	0.018046	0.034833	0.042547	0.044250	0.043604
μ_3	-0.001982	0.013853	0.035280	0.038671	0.040339	0.039671
$R_{1,1}$	0.995638	0.999636	0.030594	0.030854	0.032664	0.031616
$R_{1,2}$	-0.000188	-0.001498	0.019377	0.019421	0.022899	0.019980
$R_{2,2}$	1.001946	1.021762	0.031913	0.037566	0.038724	0.038680
$R_{1,3}$	-0.004291	-0.030309	0.018963	0.032197	0.033586	0.033113
$R_{2,3}$	0.001238	0.021554	0.018397	0.027409	0.029031	0.027764
$R_{3,3}$	0.996106	0.967894	0.030197	0.041327	0.043449	0.042494
$P_{1,1}$	0.137964	0.168808	0.081201	0.086863	0.087303	0.089219
$P_{2,2}$	0.004711	-0.001004	0.109165	0.109314	0.109444	0.114385
$P_{3,3}$	-0.058212	-0.052750	0.128347	0.128463	0.128560	0.133870
a_1	11.986857	11.991603	0.010649	0.011659	0.012774	0.012461
a_2	-1.996886	-1.998937	0.006776	0.007080	0.007853	0.007601

As for Table 2 except that $\Delta = 3$, which is the point just after the curves in Figure 3 begin to flatten.

Table 4. Demand and Supply Example, $\Delta = 11$

	Mean		Standard Deviation			
			Extrinsic		Modified	
Parameter	Extrinsic	Intrinsic	Extr Ctr	Intr Ctr	Extrinsic	Riemann
μ_1	0.006974	-0.000005	0.032410	0.033154	0.033150	0.000024
μ_2	-0.006384	0.018046	0.034833	0.042547	0.042543	0.043143
μ_3	-0.001982	0.013853	0.035280	0.038671	0.038668	0.039171
$R_{1,1}$	0.995638	0.999636	0.030594	0.030854	0.030851	0.031218
$R_{1,2}$	-0.000188	-0.001498	0.019377	0.019421	0.019419	0.019721
$R_{2,2}$	1.001946	1.021762	0.031913	0.037566	0.037563	0.038075
$R_{1,3}$	-0.004291	-0.030309	0.018963	0.032197	0.032194	0.032654
$R_{2,3}$	0.001238	0.021554	0.018397	0.027409	0.027407	0.027407
$R_{3,3}$	0.996106	0.967894	0.030197	0.041327	0.041323	0.041930
$P_{1,1}$	0.137964	0.168808	0.081201	0.086863	0.086854	0.088216
$P_{2,2}$	0.004711	-0.001004	0.109165	0.109314	0.109303	0.110615
$P_{3,3}$	-0.058212	-0.052750	0.128347	0.128463	0.128451	0.129637
a_1	11.986857	11.991603	0.010649	0.011659	0.011658	0.012298
a_2	-1.996886	-1.998937	0.006776	0.007080	0.007079	0.007500

As for Table 2 except that $\Delta = 11$, which is the smallest value such that each node in M_ϵ is connected to all other nodes.

Table 5. Moment Function Parameter Correlations

Correlation	Extrinsic: Extr Ctr	Extrinsic: Intr Ctr	Modified: Extrinsic	Modified: Riemann
	Demand and Supply Example, $\Delta = 0.9$			
$\rho(a_1, a_2)$	-0.953464	-0.959086	-0.850522	-0.970201
	Demand and Supply Example, $\Delta = 11$			
$\rho(a_1, a_2)$	-0.953464	-0.951455	-0.951455	-0.959869
	Stochastic Discount Factor Example, $\Delta = 2$			
$\rho(a_1, a_2)$	-0.538199	-0.624957	-0.619447	-0.515152
$\rho(a_1, a_3)$	-0.933849	-0.924537	-0.889809	-0.238756
$\rho(a_2, a_3)$	0.238997	0.341350	0.470240	0.442594
	Stochastic Discount Factor Example, $\Delta = 31$			
$\rho(a_1, a_2)$	-0.538199	-0.522063	-0.522063	-0.384290
$\rho(a_1, a_3)$	-0.933849	-0.931478	-0.931478	-0.991255
$\rho(a_2, a_3)$	0.238997	0.214916	0.214916	0.262880

Shown are the correlations for the parameters θ that appear in the moment functions (2) computed from V_{EC}, V_{IC}, V_{ME}, V_{MR} that were themselves computed from the MCMC chains described in Tables 2, 4, 8 and 10 for the four blocks of the table, respectively, as indicated by the headings for each block. For instance, the first entry $\rho(a_1, a_2) = -0.953464$ refers to a correlation computed from V_{EC} for the demand and supply MCMC chain described in Table 2.

Table 6. Regressions Among Standard Deviations, Demand and Supply Example

Variable: Independent	Variable: Dependent	Intercept	Slope	R^2
		$\Delta = 0.9$		
V_{EC} sdev	V_{IC} sdev	0.003445	0.996188	0.966189
V_{EC} sdev	V_{ME} sdev	0.039708	0.837890	0.829003
V_{EC} sdev	V_{MR} sdev	0.000226	1.619744	0.914482
V_{IC} sdev	V_{ME} sdev	0.036227	0.853974	0.884487
V_{IC} sdev	V_{MR} sdev	-0.005547	1.629730	0.950897
V_{ME} sdev	V_{MR} sdev	-0.060075	1.713394	0.866591
		$\Delta = 3.0$		
V_{EC} sdev	V_{IC} sdev	0.005572	0.966685	0.985068
V_{EC} sdev	V_{ME} sdev	0.007508	0.951912	0.984039
V_{EC} sdev	V_{MR} sdev	0.002167	1.027389	0.922289
V_{IC} sdev	V_{ME} sdev	0.002007	0.985043	0.999616
V_{IC} sdev	V_{MR} sdev	-0.004049	1.069157	0.947510
V_{ME} sdev	V_{MR} sdev	-0.006268	1.086251	0.949377
		$\Delta = 11.0$		
V_{EC} sdev	V_{IC} sdev	0.005572	0.966685	0.985068
V_{EC} sdev	V_{ME} sdev	0.005572	0.966685	0.985068
V_{EC} sdev	V_{MR} sdev	0.002657	0.993814	0.919170
V_{IC} sdev	V_{ME} sdev	0.000000	1.000000	1.000000
V_{IC} sdev	V_{MR} sdev	-0.003395	1.035088	0.945897
V_{ME} sdev	V_{MR} sdev	-0.003395	1.035088	0.945897

Shown in the first block are linear regressions of standard deviations from V_{EC}, V_{IC}, V_{ME}, and V_{MR} computed from the MCMC chain described in the legend for Table 2 with independent and dependent variables as indicated in the first two columns of the table. The second and third blocks are the same but for $\Delta = 3.0$ and $\Delta = 11.0$

Table 7. Regressions Among Covariances Demand and Supply Example

Variable				
Independent	Dependent	Intercept	Slope	R^2
		$\Delta = 0.9$		
V_{EC} sdev	V_{IC} sdev	-0.000021	1.018985	0.584151
V_{EC} sdev	V_{ME} sdev	-0.000075	1.640258	0.324489
V_{EC} sdev	V_{MR} sdev	-0.000081	2.483038	0.542015
V_{IC} sdev	V_{ME} sdev	-0.000044	1.975227	0.836412
V_{IC} sdev	V_{MR} sdev	-0.000031	2.457824	0.943967
V_{ME} sdev	V_{MR} sdev	0.000018	1.068931	0.832854
		$\Delta = 3.0$		
V_{EC} sdev	V_{IC} sdev	-0.000005	0.954483	0.509323
V_{EC} sdev	V_{ME} sdev	-0.000007	0.977459	0.498042
V_{EC} sdev	V_{MR} sdev	-0.000007	0.993356	0.497677
V_{IC} sdev	V_{ME} sdev	-0.000002	1.032686	0.994369
V_{IC} sdev	V_{MR} sdev	-0.000002	1.041013	0.977671
V_{ME} sdev	V_{MR} sdev	-0.000000	1.003528	0.974380
		$\Delta = 11.0$		
V_{EC} sdev	V_{IC} sdev	-0.000005	0.954483	0.509323
V_{EC} sdev	V_{ME} sdev	-0.000005	0.954483	0.509323
V_{EC} sdev	V_{MR} sdev	-0.000006	0.942636	0.484505
V_{IC} sdev	V_{ME} sdev	0.000000	1.000000	1.000000
V_{IC} sdev	V_{MR} sdev	-0.000001	1.001591	0.978439
V_{ME} sdev	V_{MR} sdev	-0.000001	1.001591	0.978439

Shown in the first block are linear regressions of covariances from V_{EC}, V_{IC}, V_{ME}, and V_{MR} computed from the MCMC chain described in the legend for Table 2 with independent and dependent variables as indicated in the first two columns of the table. The second and third blocks are the same but for $\Delta = 3.0$ and $\Delta = 11.0$

Table 8. Stochastic Discount Function Example, $\Delta = 2$

	Mean		Standard Deviation			
			Extrinsic		Modified	
Parameter	Extrinsic	Intrinsic	Extr Ctr	Intr Ctr	Extrinsic	Riemann
a_{01}	0.125950	0.130222	0.035219	0.035477	0.057211	0.085094
a_{02}	-0.008434	-0.016024	0.027108	0.028150	0.074924	0.053488
a_{03}	0.017429	0.013113	0.015427	0.016020	0.054562	0.031979
a_{04}	0.082601	0.075387	0.010530	0.012764	0.055936	0.024716
a_{05}	-0.061553	-0.074851	0.019684	0.023756	0.047917	0.043069
a_{06}	-0.036925	-0.024713	0.017924	0.021690	0.063208	0.043061
a_{07}	-0.028193	-0.010717	0.012460	0.021465	0.055240	0.040529
a_{08}	0.152953	0.164645	0.011347	0.016294	0.057191	0.033546
$b_{0,1}$	0.149272	0.159229	0.034798	0.036195	0.076915	0.072240
$b_{0,2}$	-0.246597	-0.268276	0.066741	0.070175	0.096793	0.154732
$B_{1,1}$	-0.046729	-0.034771	0.014092	0.018483	0.046557	0.027807
$B_{2,1}$	-0.058537	-0.036099	0.018411	0.029026	0.063679	0.006958
$B_{1,2}$	-0.007491	0.010769	0.019087	0.026416	0.075955	0.055285
$B_{2,2}$	-0.023266	-0.047909	0.023289	0.033908	0.082084	0.010080
$R_{0,1,1}$	0.836213	0.830120	0.026761	0.027446	0.074612	0.056217
$R_{0,1,2}$	-0.040340	-0.044094	0.010666	0.011308	0.049220	0.021801
$R_{0,2,2}$	0.993556	1.001678	0.042345	0.043117	0.108780	0.083124
$P_{1,1}$	0.551396	0.588314	0.052075	0.063836	0.102282	0.134788
$P_{2,2}$	0.099384	0.097378	0.053043	0.053081	0.106752	0.100766
a_1	-0.000000	-0.000008	0.000015	0.000016	0.000020	0.005618
a_2	-0.997967	-0.980331	0.010756	0.020659	0.043361	0.040626
a_3	-0.020725	0.013500	0.127623	0.132134	0.149616	0.272943

An MCMC chain of length 50,000 was computed using the Surface Sampling Algorithm for the SNP-ARCH likelihood (34) estimated from daily, inflation adjusted returns on the S&P500 and NASDAQ indices (including distributions) from January 1, 2010, to December 31, 2018 under moment conditions (2) as determined by (35) through (38). The prior for ρ is independent normal with location and scale the SNP-ARCH unconstrained maximum likelihood estimated parameters and standard errors. The prior for $\theta = (a_0, a_1, a_2)$ is independent normal with means $(0, -1, 0)$ and standard deviations $(1, 1, 1)$. The support conditions are normalizing sign restrictions on variance parameters and that the eigenvalues of the companion matrices for location and scale are less than one in absolute value. The chain was reduced with a stride of 10 leaving a chain of length 5,000 for computations. Means and standard deviations shown in the table for offset $\Delta = 2$, which is 1.0 larger than the smallest value of Δ for which the manifold M_ϵ is connected.

Table 9. Stochastic Discount Function Example, $\Delta = 10$

	Mean		Standard Deviation			
			Extrinsic		Modified	
Parameter	Extrinsic	Intrinsic	Extr Ctr	Intr Ctr	Extrinsic	Riemann
a_{01}	0.125950	0.118768	0.035219	0.035944	0.036786	0.036965
a_{02}	-0.008434	0.011021	0.027108	0.033367	0.034084	0.035347
a_{03}	0.017429	0.005592	0.015427	0.019446	0.020418	0.019750
a_{04}	0.082601	0.082544	0.010530	0.010530	0.011922	0.011040
a_{05}	-0.061553	-0.076661	0.019684	0.024814	0.026188	0.023665
a_{06}	-0.036925	-0.034718	0.017924	0.018060	0.019498	0.018794
a_{07}	-0.028193	-0.014846	0.012460	0.018261	0.019301	0.018503
a_{08}	0.152953	0.150794	0.011347	0.011551	0.012393	0.011898
$b_{0,1}$	0.149272	0.148000	0.034798	0.034821	0.037570	0.036086
$b_{0,2}$	-0.246597	-0.250884	0.066741	0.066879	0.067242	0.067979
$B_{1,1}$	-0.046729	-0.050522	0.014092	0.014594	0.015469	0.013500
$B_{2,1}$	-0.058537	-0.059758	0.018411	0.018451	0.019506	0.002549
$B_{1,2}$	-0.007491	-0.020881	0.019087	0.023316	0.024502	0.023297
$B_{2,2}$	-0.023266	-0.041644	0.023289	0.029668	0.032058	0.004874
$R_{0,1,1}$	0.836213	0.827857	0.026761	0.028036	0.030673	0.029207
$R_{0,1,2}$	-0.040340	-0.028008	0.010666	0.016305	0.016786	0.019185
$R_{0,2,2}$	0.993556	0.960481	0.042345	0.053734	0.054459	0.054449
$P_{1,1}$	0.551396	0.539821	0.052075	0.053346	0.054371	0.055694
$P_{2,2}$	0.099384	0.101043	0.053043	0.053069	0.055329	0.054195
a_1	-0.000000	-0.000002	0.000015	0.000015	0.000015	0.003165
a_2	-0.997967	-0.998806	0.010756	0.010789	0.011167	0.011660
a_3	-0.020725	0.008479	0.127623	0.130923	0.130910	0.139620

As for Table 8 except that $\Delta = 10$, which is the point just after the curves in Figure 4 begin to flatten.

Table 10. Stochastic Discount Function Example, $\Delta = 31$

	Mean		Standard Deviation			
			Extrinsic		Modified	
Parameter	Extrinsic	Intrinsic	Extr Ctr	Intr Ctr	Extrinsic	Riemann
a_{01}	0.125950	0.118768	0.035219	0.035944	0.035940	0.036115
a_{02}	-0.008434	0.011021	0.027108	0.033367	0.033364	0.033468
a_{03}	0.017429	0.005592	0.015427	0.019446	0.019444	0.019369
a_{04}	0.082601	0.082544	0.010530	0.010530	0.010529	0.010539
a_{05}	-0.061553	-0.076661	0.019684	0.024814	0.024812	0.025392
a_{06}	-0.036925	-0.034718	0.017924	0.018060	0.018058	0.018097
a_{07}	-0.028193	-0.014846	0.012460	0.018261	0.018259	0.018050
a_{08}	0.152953	0.150794	0.011347	0.011551	0.011550	0.011555
$b_{0,1}$	0.149272	0.148000	0.034798	0.034821	0.034818	0.034725
$b_{0,2}$	-0.246597	-0.250884	0.066741	0.066879	0.066872	0.066737
$B_{1,1}$	-0.046729	-0.050522	0.014092	0.014594	0.014592	0.014727
$B_{2,1}$	-0.058537	-0.059758	0.018411	0.018451	0.018449	0.018453
$B_{1,2}$	-0.007491	-0.020881	0.019087	0.023316	0.023314	0.023764
$B_{2,2}$	-0.023266	-0.041644	0.023289	0.029668	0.029665	0.029748
$R_{0,1,1}$	0.836213	0.827857	0.026761	0.028036	0.028033	0.028009
$R_{0,1,2}$	-0.040340	-0.028008	0.010666	0.016305	0.016304	0.016159
$R_{0,2,2}$	0.993556	0.960481	0.042345	0.053734	0.053728	0.053690
$P_{1,1}$	0.551396	0.539821	0.052075	0.053346	0.053341	0.053352
$P_{2,2}$	0.099384	0.101043	0.053043	0.053069	0.053064	0.053072
a_1	-0.000000	-0.000002	0.000015	0.000015	0.000015	0.000013
a_2	-0.997967	-0.998806	0.010756	0.010789	0.010788	0.008667
a_3	-0.020725	0.008479	0.127623	0.130923	0.130910	0.130925

As for Table 8 except that $\Delta = 31$, which is the smallest value such that each node in M_ϵ is connected to all other nodes.

Table 11. Regressions Among Standard Deviations, Stochastic Discount Function Example

Variable				
Independent	Dependent	Intercept	Slope	R^2
		$\Delta = 2.0$		
V_{EC} sdev	V_{IC} sdev	0.004282	0.996627	0.979527
V_{EC} sdev	V_{ME} sdev	0.041461	0.971254	0.769701
V_{EC} sdev	V_{MR} sdev	0.000142	2.148869	0.941703
V_{IC} sdev	V_{ME} sdev	0.037353	0.972626	0.782703
V_{IC} sdev	V_{MR} sdev	-0.007529	2.109841	0.920539
V_{ME} sdev	V_{MR} sdev	-0.050916	1.632486	0.666095
		$\Delta = 10.0$		
V_{EC} sdev	V_{IC} sdev	0.002420	1.005033	0.987339
V_{EC} sdev	V_{ME} sdev	0.003663	1.001094	0.986990
V_{EC} sdev	V_{MR} sdev	-0.000386	1.077904	0.949659
V_{IC} sdev	V_{ME} sdev	0.001260	0.995870	0.999223
V_{IC} sdev	V_{MR} sdev	-0.002833	1.067869	0.953538
V_{ME} sdev	V_{MR} sdev	-0.004076	1.069052	0.948511
		$\Delta = 31.0$		
V_{EC} sdev	V_{IC} sdev	0.002420	1.005033	0.987339
V_{EC} sdev	V_{ME} sdev	0.002420	1.005033	0.987339
V_{EC} sdev	V_{MR} sdev	0.002308	1.006940	0.986297
V_{IC} sdev	V_{ME} sdev	-0.000000	1.000000	1.000000
V_{IC} sdev	V_{MR} sdev	-0.000129	1.002270	0.999689
V_{ME} sdev	V_{MR} sdev	-0.000129	1.002270	0.999689

Shown in the first block are linear regressions of standard deviations from V_{EC}, V_{IC}, V_{ME}, and V_{MR} computed from the MCMC chain described in the legend for Table 8 with independent and dependent variables as indicated in the first two columns of the table. The second and third blocks are the same but for $\Delta = 10.0$ and $\Delta = 31.0$

Table 12. Regressions Among Covariances, Stochastic Discount Function Example

Variable				
Independent	Dependent	Intercept	Slope	R^2
		$\Delta = 2.0$		
V_{EC} sdev	V_{IC} sdev	0.000017	1.103377	0.698067
V_{EC} sdev	V_{ME} sdev	0.000071	2.383066	0.242961
V_{EC} sdev	V_{MR} sdev	0.000110	3.960890	0.457034
V_{IC} sdev	V_{ME} sdev	0.000035	3.060979	0.699095
V_{IC} sdev	V_{MR} sdev	0.000049	3.714154	0.700864
V_{ME} sdev	V_{MR} sdev	0.000020	0.850015	0.491984
		$\Delta = 10.0$		
V_{EC} sdev	V_{IC} sdev	-0.000002	1.013226	0.830676
V_{EC} sdev	V_{ME} sdev	-0.000004	1.066309	0.819366
V_{EC} sdev	V_{MR} sdev	-0.000010	1.050009	0.842409
V_{IC} sdev	V_{ME} sdev	-0.000002	1.055850	0.992882
V_{IC} sdev	V_{MR} sdev	-0.000009	0.981551	0.909796
V_{ME} sdev	V_{MR} sdev	-0.000007	0.917860	0.893262
		$\Delta = 31.0$		
V_{EC} sdev	V_{IC} sdev	-0.000002	1.013226	0.830676
V_{EC} sdev	V_{ME} sdev	-0.000002	1.013226	0.830676
V_{EC} sdev	V_{MR} sdev	-0.000002	1.014812	0.831122
V_{IC} sdev	V_{ME} sdev	0.000000	1.000000	1.000000
V_{IC} sdev	V_{MR} sdev	-0.000000	1.001103	0.999615
V_{ME} sdev	V_{MR} sdev	-0.000000	1.001103	0.999615

Shown in the first block are linear regressions of covariances from V_{EC}, V_{IC}, V_{ME}, and V_{MR} computed from the MCMC chain described in the legend for Table 8 with independent and dependent variables as indicated in the first two columns of the table. The second and third blocks are the same but for $\Delta = 10.0$ and $\Delta = 31.0$

Table 13. Curved Manifold Example, $\Delta = 0.57$

Parameter	Mean		Standard Deviation or Correlation			
			Extrinsic		Modified	
	Extrinsic	Intrinsic	Extr Ctr	Intr Ctr	Extrinsic	Riemann
μ_1	0.003030	0.001782	0.044938	0.044956	0.256304	0.045930
μ_2	0.010777	0.008102	0.046710	0.046787	0.282870	0.047894
$R_{1,1}$	0.997487	0.992473	0.030209	0.030622	0.199155	0.031385
$R_{1,2}$	-0.011216	-0.008383	0.021103	0.021293	0.133476	0.021763
$R_{2,2}$	1.029374	1.030792	0.010518	0.010614	0.066102	0.010639
θ	5.379109	5.377738	0.155378	0.155384	0.975752	0.159227
$\rho(\mu_1, \mu_2)$			-0.078107	-0.076362	-0.043754	-0.075747
$\rho(\mu_1, R_{1,1})$			-0.038925	-0.033837	-0.031125	-0.030951
$\rho(\mu_1, R_{1,2})$			-0.014263	-0.017826	-0.008660	-0.010185
$\rho(\mu_1, R_{2,2})$			-0.049502	-0.052750	-0.032077	-0.050698
$\rho(\mu_1, \theta)$			-0.034030	-0.033771	-0.025612	-0.031390
$\rho(\mu_2, R_{1,1})$			-0.000003	0.009360	-0.028539	0.010071
$\rho(\mu_2, R_{1,2})$			0.061739	0.053481	-0.018607	0.054120
$\rho(\mu_2, R_{2,2})$			-0.230838	-0.236033	-0.068083	-0.223426
$\rho(\mu_2, \theta)$			0.003121	0.003620	-0.025818	0.003372
$\rho(R_{1,1}, R_{1,2})$			-0.149040	-0.167514	0.333217	-0.161162
$\rho(R_{1,1}, R_{2,2})$			0.439249	0.407558	0.463633	0.440073
$\rho(R_{1,1}, \theta)$			0.467925	0.463035	0.475693	0.462643
$\rho(R_{1,2}, R_{2,2})$			0.762028	0.766259	0.793649	0.771083
$\rho(R_{1,2}, \theta)$			0.801051	0.792721	0.819296	0.798896
$\rho(R_{2,2}, \theta)$			0.960119	0.950296	0.936706	0.973192

The data are a simulation of the curved manifold example. An MCMC chain of length 50,000 was computed using the Surface Sampling Algorithm for the normal likelihood (39) subject to moment conditions (2) as determined by (41) and (42) The prior for ρ is independent normal with location the unconstrained maximum likelihood estimates of (39) and scale 5.0. The prior for θ is normal with mean 5.0 and standard deviations 5.0. The support conditions on R are that diagonals must be positive and θ must be positive.. The chain was reduced by eliminating repetitions due to rejections to a length of 37,269 for computations. Means and standard deviations shown in the table for offset $\Delta = 0.57$, which is the smallest value of Δ for which the manifold M_ϵ is connected.

Table 14. Curved Manifold Example, $\Delta = 3.0$

	Mean		Standard Deviation or Correlation			
			Extrinsic		Modified	
Parameter	Extrinsic	Intrinsic	Extr Ctr	Intr Ctr	Extrinsic	Riemann
μ_1	0.003030	0.001782	0.044938	0.044956	0.064382	0.044967
μ_2	0.010777	0.008102	0.046710	0.046787	0.069694	0.046818
$R_{1,1}$	0.997487	0.992473	0.030209	0.030622	0.049044	0.030692
$R_{1,2}$	-0.011216	-0.008383	0.021103	0.021293	0.034104	0.021243
$R_{2,2}$	1.029374	1.030792	0.010518	0.010614	0.015268	0.010386
θ	5.379109	5.377738	0.155378	0.155384	0.228527	0.155419
$\rho(\mu_1, \mu_2)$			-0.078107	-0.076362	-0.039940	-0.076271
$\rho(\mu_1, R_{1,1})$			-0.038925	-0.033837	-0.013112	-0.031901
$\rho(\mu_1, R_{1,2})$			-0.014263	-0.017826	0.006544	-0.012268
$\rho(\mu_1, R_{2,2})$			-0.049502	-0.052750	-0.003076	-0.053070
$\rho(\mu_1, \theta)$			-0.034030	-0.033771	-0.004220	-0.033868
$\rho(\mu_2, R_{1,1})$			-0.000003	0.009360	0.000939	0.009883
$\rho(\mu_2, R_{1,2})$			0.061739	0.053481	0.020551	0.054497
$\rho(\mu_2, R_{2,2})$			-0.230838	-0.236033	-0.046810	-0.223549
$\rho(\mu_2, \theta)$			0.003121	0.003620	0.013745	0.003537
$\rho(R_{1,1}, R_{1,2})$			-0.149040	-0.167514	0.163611	-0.162194
$\rho(R_{1,1}, R_{2,2})$			0.439249	0.407558	0.320706	0.440313
$\rho(R_{1,1}, \theta)$			0.467925	0.463035	0.329882	0.462884
$\rho(R_{1,2}, R_{2,2})$			0.762028	0.766259	0.653799	0.770204
$\rho(R_{1,2}, \theta)$			0.801051	0.792721	0.698690	0.798101
$\rho(R_{2,2}, \theta)$			0.960119	0.950296	0.886439	0.973122

As for Table 13 except that $\Delta = 3.0$, which is the point just after the curves in Figure 4 begin to flatten.

Table 15. Curved Manifold Example, $\Delta = 15.0$

	Mean		Standard Deviation or Correlation			
			Extrinsic		Modified	
Parameter	Extrinsic	Intrinsic	Extr Ctr	Intr Ctr	Extrinsic	Riemann
μ_1	0.003030	0.001782	0.044938	0.044956	0.045979	0.044963
μ_2	0.010777	0.008102	0.046710	0.046787	0.049665	0.046814
$R_{1,1}$	0.997487	0.992473	0.030209	0.030622	0.036357	0.030688
$R_{1,2}$	-0.011216	-0.008383	0.021103	0.021293	0.024844	0.021239
$R_{2,2}$	1.029374	1.030792	0.010518	0.010614	0.010586	0.010383
θ	5.379109	5.377738	0.155378	0.155384	0.158211	0.155382
$\rho(\mu_1, \mu_2)$			-0.078107	-0.076362	-0.017182	-0.076274
$\rho(\mu_1, R_{1,1})$			-0.038925	-0.033837	-0.003856	-0.031906
$\rho(\mu_1, R_{1,2})$			-0.014263	-0.017826	-0.022850	-0.012280
$\rho(\mu_1, R_{2,2})$			-0.049502	-0.052750	-0.035418	-0.053089
$\rho(\mu_1, \theta)$			-0.034030	-0.033771	-0.033630	-0.033885
$\rho(\mu_2, R_{1,1})$			-0.000003	0.009360	0.011497	0.009885
$\rho(\mu_2, R_{1,2})$			0.061739	0.053481	0.028671	0.054493
$\rho(\mu_2, R_{2,2})$			-0.230838	-0.236033	-0.083846	-0.223585
$\rho(\mu_2, \theta)$			0.003121	0.003620	0.016311	0.003531
$\rho(R_{1,1}, R_{1,2})$			-0.149040	-0.167514	-0.047995	-0.162304
$\rho(R_{1,1}, R_{2,2})$			0.439249	0.407558	0.186387	0.440271
$\rho(R_{1,1}, \theta)$			0.467925	0.463035	0.209996	0.462847
$\rho(R_{1,2}, R_{2,2})$			0.762028	0.766259	0.448719	0.770156
$\rho(R_{1,2}, \theta)$			0.801051	0.792721	0.519493	0.798059
$\rho(R_{2,2}, \theta)$			0.960119	0.950296	0.797676	0.973115

As for Table 13 except that $\Delta = 15.0$, which is the smallest value of Δ such that each node in M_ϵ is connected to all other nodes.

Table 16. Regressions Among Standard Deviations, Curved Manifold Example

Variable				
Independent	Dependent	Intercept	Slope	R^2
		$\Delta = 0.57$		
V_{EC} sdev	V_{IC} sdev	0.000201	0.998685	0.999993
V_{EC} sdev	V_{ME} sdev	-0.004274	6.279022	0.998619
V_{EC} sdev	V_{MR} sdev	0.000080	1.024291	0.999985
V_{IC} sdev	V_{ME} sdev	-0.005557	6.287697	0.998756
V_{IC} sdev	V_{MR} sdev	-0.000126	1.025643	0.999998
V_{ME} sdev	V_{MR} sdev	0.000844	0.162919	0.998793
		$\Delta = 3.0$		
V_{EC} sdev	V_{IC} sdev	0.000201	0.998685	0.999993
V_{EC} sdev	V_{ME} sdev	0.001779	1.458109	0.999163
V_{EC} sdev	V_{MR} sdev	0.000132	0.999614	0.999985
V_{IC} sdev	V_{ME} sdev	0.001482	1.460118	0.999292
V_{IC} sdev	V_{MR} sdev	-0.000069	1.000932	0.999997
V_{ME} sdev	V_{MR} sdev	-0.001049	0.685046	0.999337
		$\Delta = 15.0$		
V_{EC} sdev	V_{IC} sdev	0.000201	0.998685	0.999993
V_{EC} sdev	V_{ME} sdev	0.002700	1.001903	0.998382
V_{EC} sdev	V_{MR} sdev	0.000135	0.999362	0.999985
V_{IC} sdev	V_{ME} sdev	0.002494	1.003311	0.998565
V_{IC} sdev	V_{MR} sdev	-0.000066	1.000680	0.999996
V_{ME} sdev	V_{MR} sdev	-0.002478	0.995999	0.998667

Shown in the first block are linear regressions of standard deviations from V_{EC}, V_{IC}, V_{ME}, and V_{MR} computed from the MCMC chain described in the legend for Table 13 with independent and dependent variables as indicated in the first two columns of the table. The second and third blocks are the same but for $\Delta = 3.0$ and $\Delta = 15.0$

Table 17. Regressions Among Covariances, Curved Manifold Example

Variable				
Independent	Dependent	Intercept	Slope	R^2
		$\Delta = 0.57$		
V_{EC} sdev	V_{IC} sdev	-0.000000	1.000302	0.999944
V_{EC} sdev	V_{ME} sdev	0.000985	40.107354	0.985408
V_{EC} sdev	V_{MR} sdev	0.000004	1.050841	0.999935
V_{IC} sdev	V_{ME} sdev	0.001009	40.078723	0.984650
V_{IC} sdev	V_{MR} sdev	0.000004	1.050508	0.999961
V_{ME} sdev	V_{MR} sdev	-0.000015	0.025815	0.985080
		$\Delta = 3.0$		
V_{EC} sdev	V_{IC} sdev	-0.000000	1.000302	0.999944
V_{EC} sdev	V_{ME} sdev	0.000114	1.861595	0.982616
V_{EC} sdev	V_{MR} sdev	0.000002	1.002748	0.999959
V_{IC} sdev	V_{ME} sdev	0.000115	1.860169	0.981758
V_{IC} sdev	V_{MR} sdev	0.000002	1.002431	0.999986
V_{ME} sdev	V_{MR} sdev	-0.000052	0.529139	0.982031
		$\Delta = 15.0$		
V_{EC} sdev	V_{IC} sdev	-0.000000	1.000302	0.999944
V_{EC} sdev	V_{ME} sdev	0.000016	0.705615	0.962668
V_{EC} sdev	V_{MR} sdev	0.000002	1.002291	0.999958
V_{IC} sdev	V_{ME} sdev	0.000017	0.705142	0.962014
V_{IC} sdev	V_{MR} sdev	0.000002	1.001975	0.999986
V_{ME} sdev	V_{MR} sdev	-0.000005	1.367140	0.962229

Shown in the first block are linear regressions of covariances from V_{EC}, V_{IC}, V_{ME}, and V_{MR} computed from the MCMC chain described in the legend for Table 13 with independent and dependent variables as indicated in the first two columns of the table. The second and third blocks are the same but for $\Delta = 3.0$ and $\Delta = 15.0$

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